What they don't teach you about integration at school

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1. Introduction

As a mathematician involved in teaching students whose abilities range from barely numerate to MSc level, I am frequently concerned by the lack of basic understanding exhibited by pupils regarding the subject of integration. Invariably the most practical way of introducing the subject of integration is by thinking of it as "anti-differentiation", so that the problem is to pick a function which "when differentiated will give what you first thought of". This of course may be allied to the idea of the area under a curve to give the student a conceptual feel for the subject Usually when simple rules for the integration of powers, trigonometric functions, exponentials, logarithms and so on have been mastered, the next step is to introduce the standard methods for proceeding. These basically break down into the following classes:

(i) Integration by substitution.
(ii) Integration by parts.
(iii) "Logarithmic" integration.
(iv) Integration by reduction formula.

The pupil then proceeds happily to apply these to the standard examples, hopefully gaining the proficiency to answer any examination question which might come his way. As we will see shortly however there are many examples where a rigorous application of these rules is not the best way of proceeding, and some lateral thinking is called for.

In most calculus and analysis courses the subject of integration is dealt with after differentiation has been mastered and the idea that any suitably "well-behaved" function may be differentiated merely by applying a number of standard rules has become well ingrained It frequently comes as a shock to students therefore to find that there are functions for which no "anti-derivative" (or more properly primitive) may be found. At first they dismiss their inability to find a primitive for $\sin x/x$ (to name one of many examples) as a lack of knowledge of "more complicated" integration rules. When they are told that there is *no* function which when differentiated will give $\sin x/x$, reactions vary from doubting that the non-existence of such a function could ever be proved, to a deep sense of disquiet that integration has somehow failed them in not providing answers to every problem. Many also have the initial reaction that the integral is in some way meaningless if no primitive can be found, a worry which may quickly be dispelled by drawing the relevant curve and convincing the doubter that the area under the curve exists in the normal way. The idea that differentiation in general maps known functions to known functions has been inbuilt *via* experience of differentiation, and it is therefore hard to understand that the mapping is not one-to-one in any sense, the image under differentiation being a proper subspace of known functions.

When some understanding of these admittedly rather deep mathematical concepts is gained, further confusion is caused by the revelation that some integrands without primitives can nevertheless be integrated between certain limits. Methods for performing such integrations are rarely mentioned in standard courses, which I feel is a pity as they call not for parrot-style regurgitation of standard integration rules, but some real mathematical insight and lateral thinking.

Postponing for a moment the discussion of strategies for definite integration of functions with no primitive, it is relevant to mention a few cases where although a primitive exists and can be found by standard methods, a quicker solution is possible The first and most obvious example of this sort is an integral like

$$\int_{-\pi/4}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)}\, dx.$$

Although it takes but a moment's thought to identify the integrand as odd and the integral as zero, try giving this to an average class and watching them fill the page with partial fraction expansions. On a similar theme, accuracy in integration by parts is often put to the test when students are faced with

$$I = \int e^{ax} \cos bx\, dx.$$

How much easier it is to exercise a little "complex" thinking and write

$$I = \text{Re} \int e^{ax+ibx}\, dx = \text{Re}\left[\frac{e^{ax+ibx}}{a+ib}\right] = \left[\frac{a\cos bx + b \sin bx}{a^2+b^2}\right] e^{ax}.$$

Similar lines of reasoning may be used to extend the "cover-up rule" for partial fractions to handle quadratic factors:

$$I = \int \frac{dx}{x(x^2+1)} = \int \left[\frac{1}{x} - \frac{1}{2(x-i)} - \frac{1}{2(x+i)}\right] dx$$

$$= \ln x - \tfrac{1}{2}\ln(x+i) - \tfrac{1}{2}\ln(x-i) = \ln\left(\frac{x}{\sqrt{x^2+1}}\right)$$

and it is possible to think up many other examples where a little thought can save a lot of work (and widen the student's general understanding).

Moving on now to examples that involve functions where no primitive exists, we may break down the available methods into the following classes:

(i) Methods relying on symmetry properties of the integrand and limits.
(ii) Multi-dimensional methods
(iii) Differentiation.
(iv) Other ad hoc methods

I have deliberately left out contour integration, transform methods and other more advanced ideas such as series summation in this list, with the idea of considering only methods which would be available to a typical first-year student or an advanced A- or S-level pupil. We discuss each of these methods in turn below.

A beautiful example of the use of symmetry properties occurs in the evaluation of the integral

$$I=\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Attempts to find a primitive are doomed to failure, but on making the substitution $x=\pi/2-y$ we find that

$$I=\int_0^{\pi/2}\frac{\sqrt{\cos y}}{\sqrt{\sin y}+\sqrt{\cos y}}\,dy$$

and hence

$$I+I=\int_0^{\pi/2}\left(\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}+\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right)dx=\int_0^{\pi/2}dx=\frac{\pi}{2}$$

so that $I=\pi/4$.

A similar strategy may be employed to evaluate many trigonometrical integrals where the limits of integration range from zero to some multiple of π. To give just one further example, we use the substitution $x=\pi-y$ to show that

$$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=\pi\int_0^{\pi}\frac{\sin y}{1+\cos^2 y}\,dy-I$$

$$=-\pi\int_0^{\pi}\frac{d(\cos y)}{1+\cos^2 y}-I=\frac{\pi^2}{2}-I$$

and hence $I=\pi^2/4$.

As far as multi-dimensional methods are concerned, the classic example of evaluating

$$I=\int_{-\infty}^{\infty}e^{-x^2}\,dx=\sqrt{\pi}$$

by considering

$$I^2=\left(\int_{-\infty}^{\infty}e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty}e^{-y^2}\,dy\right)=\int_{\mathbb{R}^2}e^{-(x^2+y^2)}\,dx\,dy$$

$$=\int_0^{2\pi}\int_0^{\infty}r e^{-r^2}\,dr\,d\theta=-2\pi\left[\frac{e^{-r^2}}{2}\right]_0^{\infty}=\pi$$

still astounds students every year. Similar arguments may be used in more complicated coordinate systems to prove a whole host of results.

Methods involving differentiation are frequently invaluable in cases where no primitive exists. One of the commonest examples is in the evaluation of

$$I=\int_0^{\infty} e^{-a^2x^2}\cos 2bx\,dx.$$

Differentiation of the integral with respect to b allows us to prove easily that

$$\frac{dI}{db}+\frac{2b}{a^2}I=0$$

and thus $I=K\exp(-b^2/a^2)$. The exponential integral evaluated above provides a boundary condition for K when $b=0$, and we finally conclude that

$$I=\frac{\sqrt{\pi}}{2a}e^{-b^2/a^2}.$$

On a similar theme, consider the function

$$G(y)=\int_0^{\infty} e^{-xy}\frac{\sin x}{x}\,dx$$

for $y>0$. Differentiation with respect to y and a simple integration shows that

$$G'(y)=\frac{-1}{1+y^2}\quad\text{and}\quad G(y)=-\tan^{-1}(y)+C$$

where C is a constant which can be evaluated as $\pi/2$ by noting that the integral form of $G(y)\to 0$ as $y\to\infty$. Thus

$$\int_0^{\infty} e^{-xy}\frac{\sin x}{x}\,dx=\frac{\pi}{2}-\tan^{-1}y$$

and in particular

$$\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

Finally, as an example to show how devious it is necessary to be in some cases, consider the integral (encountered in the theory of small elastic displacements of thin plates)

$$\int_{-\alpha}^{\alpha}\frac{\cos\lambda_m\theta}{(\cos\theta)^{2+\lambda_m}}\,d\theta$$

where the eigenvalues λ_m satisfy the relationship

$$\lambda_m=\frac{(2m-1)\pi}{2\alpha}-1$$

Evaluation of this integral by normal means is very difficult, but if we begin by noting that in the special case $\alpha = \pi/4$ the integral reduces to

$$\int_{-\pi/4}^{\pi/4} \frac{\cos(4m-3)\theta}{(\cos\theta)^{4m-1}}\,d\theta = \frac{(-1)^{m+1}2^{2m}}{(4m-2)}$$

(by elementary trigonometrical relationships) then we may proceed as follows: Consider the function

$$I(\lambda_m, \alpha) = \int_{-\alpha}^{\alpha} \frac{\cos\lambda_m\theta}{(\cos\theta)^{2+\lambda_m}}\,d\theta$$

where α and λ_m are regarded as *independent* variables. By the fundamental theorem of calculus

$$\frac{\partial I}{\partial\alpha} = \frac{2\cos\lambda_m\alpha}{(\cos\alpha)^{2+\lambda_m}}$$

but from the definition of λ_m we have

$$\cos\lambda_m\alpha = \sin(m\pi - \alpha) = (-1)^{m+1}\sin\alpha.$$

Consequently

$$\frac{\partial I}{\partial\alpha} = \frac{2(-1)^{m+1}\sin\alpha}{(\cos\alpha)^{2+\lambda_m}}$$

Integrating this with respect to α gives

$$I = \frac{2(-1)^{m+1}(\cos\alpha)^{-1-\lambda_m}}{(1+\lambda_m)} + f(\lambda_m)$$

where f is an arbitrary function, but using the special case $\alpha = \pi/4$ mentioned above we soon see that $f(\lambda_m) = 0$, so that

$$I = \frac{2(-1)^{m+1}(\cos\alpha)^{-1-\lambda_m}}{(1+\lambda_m)}$$

The rigorous justification of this seemingly dubious method is not too hard, and is left as an exercise for the interested reader. It is worth mentioning however that the idea of regarding related variables as independent is a commonplace one, used for example when employing Lagrange multipliers for constrained minimisation problems.

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