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# Film cooling effectiveness for subsonic slot injection into a cross flow

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**Summary.** A model is developed that allows the prediction of the film cooling effectiveness produced by slot injection into a uniform cross flow The model relies on the fact that when the slot pressure exceeds the cross flow pressure by a small amount only so that injection is weak, the resulting small parameter may be exploited to solve the flow problem. The energy equation for the flow may then be solved to determine how much protection cold gas injection gives to the wall downstream of a slot. Although the leading order energy equation must be solved numerically, a simple asymptotic expression may also be derived to allow predictions of heat transfer at large distances from the injection slot.

## 1 Introduction

In modern jet engines turbine entry temperatures are routinely higher than the melting temperature of the material of which the turbine blades are composed. It is thus essential that the blade surfaces are protected. The commonest way of providing such protection is to employ various forms of "film cooling" wherein a film of cooling air is injected into the flow through small slots or holes in the surface of the turbine blades Practial details of the general problem were considered in [1] with specific reference to jet engines; injection problems similar to this also occur in many other industrial processes such as film de-icing and chemical mixing.

When film cooling is employed, downstream blade protection is increased as the mass flow of injected air increases from zero, though evidently for sufficiently high injection rates the cooling jets will not remain close to the wall that they are designed to protect. Much interest is therefore centered around the low injection rate problem where the pressure in the slot or hole exceeds that of the free stream by only a small amount.

In the current study we wish to employ asymptotic techniques to assess the ability of two-dimensional slot film cooling to provide thermal protection for a blade surface downstream of a film cooling slot. Below we show that it is possible to do this by extending previous models to incorporate the effects of varying temperature.

The basis for a simple model of low injection rate slot film cooling that agreed well with experimental results was established by Fitt et al [2], and the reader is referred to that paper for details of other work in this area. For a review of earlier experimental and semi-empirical work in the subject, see [3]. The analogous problem for the case of suction (when the injection and free stream total pressures are equal) has been studied by Dewynne et al. [4] using hodograph methods When the total pressures are unequal however (as in the case that we wish to consider) such methods are no longer applicable.

## 2 Mathematical modelling

In order to derive equations that allow film cooling effectiveness predictions to be made, we consider the geometry shown schematically in Fig. 1 The injection is driven by a small overpressure, so that deep within the slot (of width $L$) we assume that

$$p_s = p_\infty + \frac{1}{2}\varrho_\infty U_\infty^2 \varepsilon^2$$

where $\varepsilon \ll 1$ defines the small parameter in the problem and $p_\infty$, $\varrho_\infty$ and $U_\infty$ are respectively the pressure, density and speed of the undisturbed cross flow

Assuming that injection takes place into an incompressible irrotational free stream, the equations of motion are

$$\nabla^2\psi_o = 0 \qquad (y > S(x)) \tag{1}$$

$$\nabla^2\psi = 0 \qquad (y < S(x)) \tag{2}$$

$$\left(\frac{\partial}{\partial t} + q \quad \nabla\right)(y - S(x)) = 0 \qquad (y = S(x)) \tag{3}$$

$$\psi_x = 0 \qquad (y = 0,\ x > 1) \tag{4}$$

$$u\theta_x + v\theta_y = \frac{k}{\varrho_\infty c_p}(\theta_{xx} + \theta_{yy}) \tag{5}$$

and are subject to the additional requirement that the pressure is continuous across $y = S(x)$

In the above equations, $\psi_o$ and $\psi$ denote the stream functions for the main stream and injected flows respectively, the velocity $q$ is given by $q = ue_x + ve_y$ where $e_x$ and $e_y$ denote unit vectors in

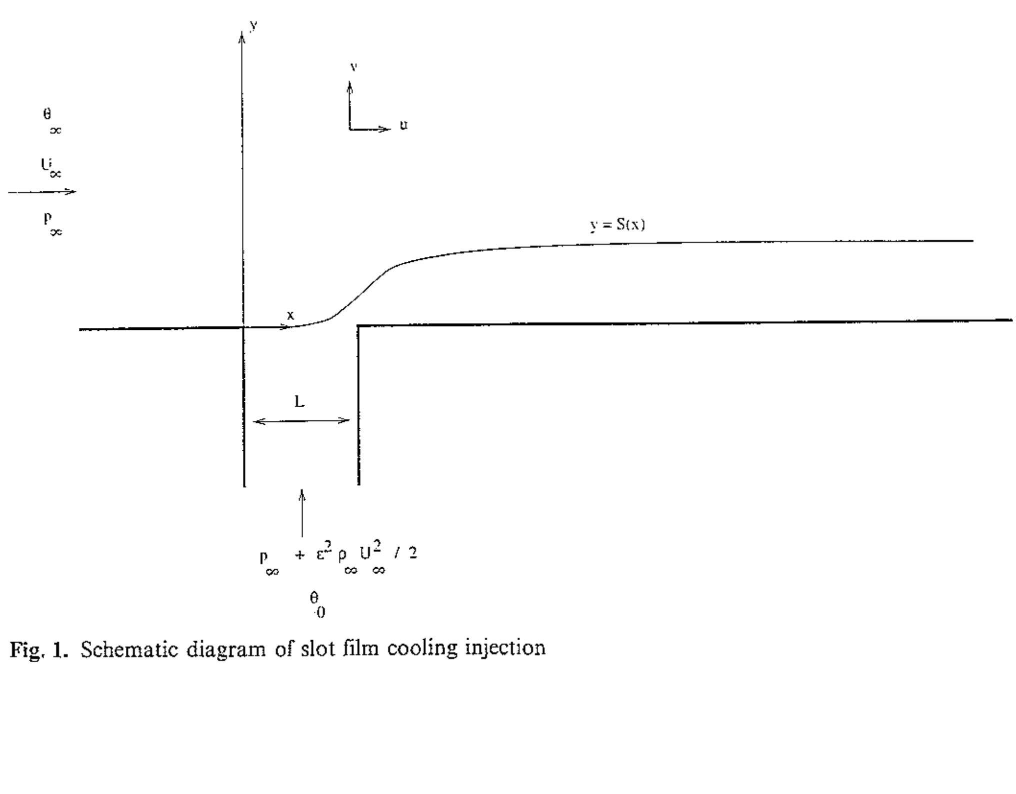


Fig. 1. Schematic diagram of slot film cooling injection

the $x$ and $y$ directions respectively. $k$, $\varrho_\infty$ and $c_p$ are respectively the constant thermal conductivity, density and specific heat at constant pressure of the injected gas (assumed for simplicity to be the same as the free stream gas; if the values were different then a few obvious changes could be made to the model). Finally, it is assumed that the main stream and injected flows are separated by the (unknown) streamline $y = S(x)$, whilst the variable $\theta$ denotes temperature.

*2.1 The flow problem*

Since the heat transfer problem decouples from the flow problem, we solve the flow problem first and then consider the heat transfer. A full discussion of the solution to the flow problem (using a slightly different method) is given in [2], but for completeness we briefly summarise the more important details. We consider only the case of steady flow (the unsteady problem may also be analysed but the details are much more complicated). Using standard thin aerofoil theory assumptions (see, for example [5]) and assuming that the injected flow layer may be represented by a distribution of sources of strength $g(t)$ (to be determined), we note that if the pressure is to be continuous across $y = S(x)$, then the order of magnitude of the perturbations to the pressure in the injected layer dictates that the stream function for the free stream flow must have the form

$$\psi_o = U_\infty y + \varepsilon^2 \frac{U_\infty}{\pi} \int_0^\infty g(t) \tan^{-1}\left(\frac{y}{x-t}\right) dt + o(\varepsilon^2) \tag{6}$$

if (1) is to be satisfied. The dividing streamline $y = S(x)$ must therefore be of height $O(L\varepsilon^2)$. Using the steady version of (3) applied on $y = 0$, we find that $g(t) = \bar{S}_{\bar{x}}$ where non dimensional variables have been defined by $x = L\bar{x}$ and $S(x) = L\varepsilon^2\bar{S}(\bar{x})$. Using Bernoulli's equation, the (non dimensional) pressure $\bar{p}$ in the outer flow on $y = 0$ may now be calculated, giving

$$\bar{p} = \frac{p_\infty}{\varrho_\infty U_\infty^2} - \frac{\varepsilon^2}{\pi} \fint_0^\infty \frac{\bar{S}'(t)}{\bar{x} - t}\, dt, \tag{7}$$

the bar on the integral denoting the Cauchy Principal Value.

In the injected layer downstream of the slot we non-dimensionalise by setting $x = Lx^*$, $y = L\varepsilon^2 y^*$, $S = L\varepsilon^2 S^*$, $\psi = LU_\infty \varepsilon^3 \psi^*$, $p = \varrho_\infty U_\infty^2 p^*$. It is apparent that, to produce pressure perturbations of the correct order of magnitude, the horizontal velocity in the downstream film region must be $O(\varepsilon U_\infty)$ (and consequently a vortex sheet separates the main stream and injected flows). The injectant mass flow is thus of order $LU_\infty \varepsilon^3$, and accordingly we non-dimensionalise by setting $M = \varepsilon^3 LU_\infty M^*$.

Solving (2) to leading order and using (4) we find that (assuming that $\psi^* = 0$ on $y^* = 0$ and $\psi^* = M^*$ on $y^* = S^*(x^*)$)

$$\psi^* = \frac{M^* y^*}{S^*}.$$

Bernoulli's equation may now be used to relate the pressure in the injected layer to the slot pressure, giving

$$p^* = \frac{p_\infty}{\varrho_\infty U_\infty^2} + \frac{\varepsilon^2}{2} - \frac{M^{*2}\varepsilon^2}{2S^{*2}}. \tag{8}$$

It remains only to consider the region directly above the slot. In the slot, since the mass flow is $O(\varepsilon^3 L U_\infty)$, the vertical velocity must be of order $\varepsilon^3 U_\infty$. It is assumed that these orders of magnitude apply right up to the top of the slot; naturally there will be small regions near to the slot leading and trailing edges where this will not apply. In the slot the (non-dimensional) pressure is thus given by

$$p^* = \frac{p_\infty}{\varrho_\infty U_\infty^2} + \frac{\varepsilon^2}{2} + O(\varepsilon^6). \tag{9}$$

To close the model continuity of pressure across $y = S(x)$ is invoked. The pressures given by (7), (8) and (9) must therefore be equal. Dropping all of the bars and stars for simplicity, we find that $S(x)$ is determined by the nonlinear singular integro differential equation

$$-\frac{1}{\pi} \int_0^\infty \frac{S'(t)}{t - x}\, dt = \begin{cases} -\dfrac{1}{2} & (0 < x < 1) \\ -\dfrac{1}{2} + \dfrac{M^2}{2S^2} & (1 < x < \infty) \end{cases} \tag{10}$$

which must be solved subject to the conditions $S(0) = 0$ (since separation is from the upstream edge of the slot) and $S'(0) = 0$ (since separation must be tangential to the wall upstream of the slot as the total pressure of the main stream flow exceeds that of the injected flow). The mass flow $M$ may be removed from the problem by setting $S(x) = M^{2/3} T(x)$ and inverting, subject to the condition $T'(0) = 0$, to give

$$T'(x) = \frac{\sqrt{x}}{2\pi} \int_1^\infty \frac{d\xi}{T^2(\xi)\, \sqrt{\xi}(\xi - x)}. \tag{11}$$

The asymptotic properties of (11) may easily be established; we find that

$$\begin{aligned} T(x) &\sim x^{3/2} \qquad (x \to 0) \\ T(x) &\sim T(\infty) - \frac{T^4(\infty)}{\pi x} \qquad (x \to \infty) \end{aligned} \tag{12}$$

The equation may easily be cast into a form suitable for numerical solution. By integrating once again and using the boundary conditions, we obtain

$$T(x) = \frac{1}{2\pi} \int_1^\infty T^{-2}(\xi) \left[ -2\sqrt{\frac{x}{\xi}} + \log\left( \frac{\sqrt{\xi} + \sqrt{x}}{|\sqrt{\xi} - \sqrt{x}|} \right) \right] d\xi \tag{13}$$

Since (13) contains no singular integrals or derivatives, it may conveniently be solved using direct iteration. Full details of the scheme employed are given in [2], but it transpires that, provided some under-relaxation is used, it is possible to use an exponential grid in order to extend the mesh to suitably large values of $x$, giving the mass flow as

$$M = T^3(\infty) \sim 1.106$$

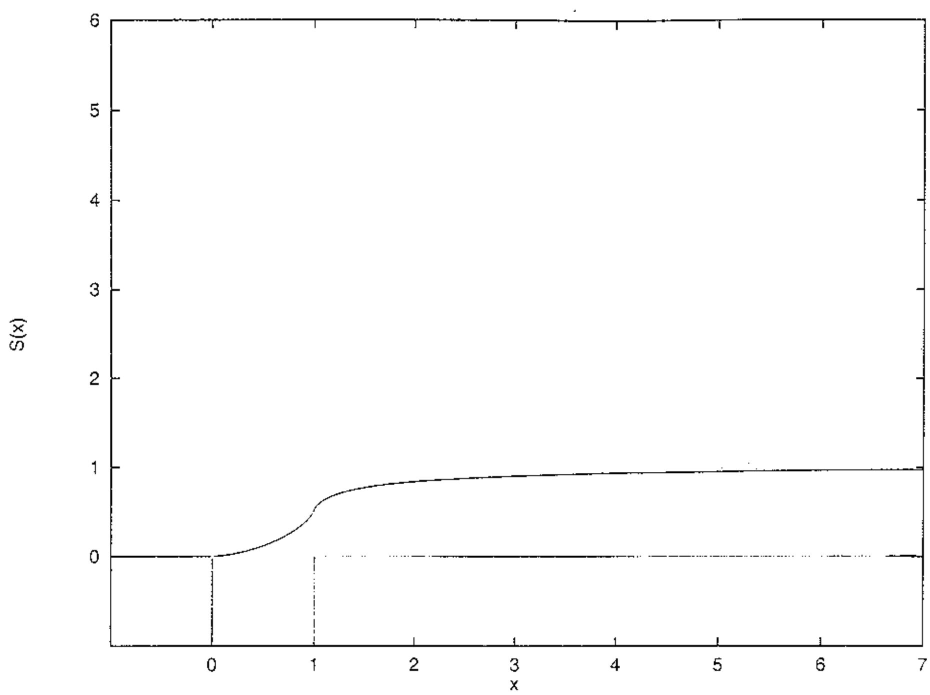


**Fig. 2.** Streamline $S(x)$ dividing mainstream and injected flow

Figure 2 shows the function $T(x)$. Perhaps the most notable feature of flow is the 'lid' effect wherein the greater total pressure head of the outer flow and consequent tangential separation from $x = 0$ acts to effectively cover most of the slot, the injected flow only escaping from the rear of the slot in any quantity.

## *2.2 The thermal problem*

Having solved the flow problem and determined $T(x)$, we now turn to our main concern, namely the heat transfer characteristics of the flow. The gas within the slot is assumed to have temperature $\theta_0$ and the Eq. (5) applies in the injectant region downstream of the slot, whilst in the mainstream flow we assume that $\theta = \theta_\infty$ Non-dimensionalising by setting $\theta = \theta_0 + \theta^*(\theta_\infty - \theta_0)$ and using the scalings given above for the injectant region downstream of the slot, we find that

$$\psi^*_{y^*}\theta^*_{x^*} - \psi^*_{x^*}\theta^*_{y^*} = \lambda(\varepsilon^4\theta^*_{x^*x^*} + \theta^*_{y^*y^*}) \tag{14}$$

where the size of

$$\lambda = \frac{k}{L\varrho_\infty c_p U_\infty \varepsilon^5}$$

determines which terms are dominant in (14).

It is necessary to specify initial and boundary conditions for (14). Most previous studies have characterised the efficiency of the film cooling by using the 'film cooling effectiveness' $\eta$ defined by

$$\eta = \frac{\theta_\infty - \theta_{aw}}{\theta_\infty - \theta_0}$$

where $\theta_{aw}$ is the "adiabatic wall temperature" (the wall temperature in the presence of a perfectly insulated wall) In terms of the non-dimensional variables this gives $\eta = 1 - \theta^*(x, 0)$. One obvious advantage of this definition is that it gives "worst case" estimates. Although in most realistic circumstances the blade surface $y^* = 0$ is unlikely to be perfectly insulated (owing to internal blade cooling etc.), we assume that $\theta^*_{y^*}(x^*, 0) = 0$ It should be noted that a more general boundary condition involving a heat transfer coefficient could easily be incorporated into the model On $y^* = S^*(x^*)$ we have $\theta^* = 1$, and we choose to additionally make the simplification that $\theta^* = 0$ at $x^* = 1$ This amounts to assuming that the hot gas of the main stream has no effect on the injected gas until it reaches the downstream edge of the slot and is justified by the existence of the "lid" effect referred to earlier

As far as the size of $\lambda$ is concerned, there are three cases to consider·

(a) When $\lambda \ll 1$ convection dominates the heat transfer, the film cooling effect is pronounced and the downstream surface is well insulated from the hot free stream A boundary layer of width $\sqrt{\lambda}$ determines the details of the heat transfer close to the surface In practical circumstances $\lambda$ is often substantially less than one, but not asymptotically small Even if it is, it must be remembered that the vortex sheet separating the injected and main stream flows is unstable and is likely to break up far downstream of the slot

(b) When $\lambda \gg 1$ diffusion dominates and heat transfer takes place rapidly To lowest order (14) gives

$$\theta^* = y^*A(x^*) + B(x^*)$$

and the boundary conditions on $y^* = 0$ and $y^* = S^*(x^*)$ give simply $\theta^* = 1$ The condition at $x^* = 1$ is satisfied via a boundary layer in $x^*$, but the details are immaterial, for in this case the film cooling effect is negligible and henceforth we ignore this parameter regime.

(c) When $\lambda = O(1)$ appreciable film cooling occurs over a number of slot widths downstream of injection. This is the most interesting case, and the one that will be analysed

For the case $\lambda \sim 1$, we use a simple numerical scheme to solve (14). Dropping the stars and setting $S = TM^{2/3}$, we find that the problem to be solved for $x \geqq 1$, $0 \leqq y \leqq M^{2/3}T(x)$ is

$$\frac{1}{T}\theta_x + \frac{yT'}{T^2}\theta_y = \alpha\theta_{yy} \tag{15}$$

with

$$\theta_y(x, 0) = 0, \quad \theta(1, y) = 0, \quad \theta(x, M^{2/3}T(x)) = 1 \tag{16}$$

where $\alpha = \lambda/M^{1/3}$. Using a rectangular grid with vertical spacing $p$ and horizontal spacing $h$, we denote the value of the numerical approximation to $\theta$ at $(ih, jp)$ by $\theta_{i,j}$. The simplest discretisations to use are

$$\theta_x \simeq \frac{\theta_{i+1,j} - \theta_{i,j}}{h}, \quad \theta_{yy} \simeq \frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{p^2}$$

and a central difference for $\theta_y$ given by

$$\theta_y \simeq \frac{\theta_{i,j+1} - \theta_{i,j-1}}{2p}$$

Denoting $T(x_i)$ by $T_i$, we use a forward difference for the derivative of $T$(since this function is already known). This gives the simple explicit finite difference scheme

$$\begin{aligned}\theta_{i+1,j} = \theta_{i,j+1}\left(\frac{h\alpha T_{i+1}}{p^2} - \frac{y_j(T_{i+1} - T_i)}{2pT_{i+1}}\right) + \theta_{i,j}\left(1 - \frac{2h\alpha T_{i+1}}{p^2}\right)\\ + \theta_{i,j-1}\left(\frac{h\alpha T_{i+1}}{p^2} + \frac{y_j(T_{i+1} - T_i)}{2pT_{i+1}}\right)\end{aligned} \tag{17}$$

The condition at $y = 0$ is dealt with using the usual fictitious points to yield

$$\theta_{i+1,0} = \theta_{i,1}\frac{2h\alpha T_{i+1}}{p^2} + \theta_{i,0}\left(1 - \frac{2h\alpha T_{i+1}}{p^2}\right) \tag{18}$$

Some standard analysis shows that linear stability is guaranteed as long as

$$\bar{\mu} = \frac{h\alpha T_{i+1}}{p^2} \leqq \frac{1}{2}.$$

Since $T(x) \leqq M^{1/3} \sim 1.011$ for all $x$, in practice stability is assured as long as the ratio

$$\mu = \frac{h\alpha}{p^2}$$

is taken to be fractionally less than 1/2.

The numerical method presented above must be used in conjunction with known numerical results for $T(x)$. Tests showed that provided enough mesh points were used for the preliminary calculation of $T(x)$, linear interpolation provided a satisfactory way of determining the values $T_i$ when required. The method given by Eqs. (17) and (18) is extremely simple, but numerical comparisons with a range of other standard methods (Crank-Nicholson, method of lines etc.) showed it to be satisfactorily accurate provided enough points were used.

As well as numerical results, asymptotic expressions may be determined to approximate the film cooling effectiveness for large values of $x$. Using (12) in (15), we find that to leading order the second term in Eq. (15) is $O(1/x^2)$ for large $x$. Correct to $O(1/x)$, therefore, the problem that must be solved is

$$\theta_x = \alpha\left(T_\infty - \frac{T_\infty^4}{\pi x}\right)\theta_{yy} \tag{19}$$

(where $T_\infty = T(\infty)$) with $\theta_y = 0$ on $y = 0$ and $\theta = 1$ on $y = M^{2/3}(T_\infty - T_\infty^4/\pi x)$. An initial condition must also be satisfied, and this will occur in the form of an (as yet unknown) matching condition. Although this is not a simple problem to solve, some simple asymptotic estimates may be determined by relaxing the condition on $y = S(x)$ slightly and instead imposing only that $\theta = 1$ on $y = M^{2/3}T_\infty$. Although this is a simplification, it is consistent with the thin aerofoil theory assumptions that have already been employed. The solution to (19) is then given by

$$\theta = 1 - K\exp\left(\frac{\pi T_\infty \alpha}{4S_\infty^2}[\pi(1 - x) + T_\infty^3 \log x]\right)\cos\left(\frac{\pi y}{2S_\infty}\right) \tag{20}$$

To determine the constant $K$ in (20), a boundary layer problem must be solved for small $x$ to determine a matching condition. Rather than involve ourselves in the complications that this entails, however, a simple formal conservation argument may be employed to assert that since at $x = 1$ the integral of $\theta$ between 0 and $S_\infty$ must be $S_\infty/2$ (a fact easily confirmed if both sides of Eq. (10) are multiplied by $S'(x)$ and integrated over $[0, \infty)$), the downstream asymptotic temperature given by Eq. (20) must satisfy the same condition, thus giving $K = \pi/4$. The expression (20) then allows useful asymptotic estimate of the film cooling effectiveness to be made.

## 3 Results and conclusions

All of the results presented below used the following parameters: The flow problem was solved using a 200 point exponential grid with $x_0 = 1$ and

$$x_n = 1 + \nu\, dx_0 \left(\frac{1 - \nu^{n-1}}{1 - \nu}\right)$$

where the initial grid spacing $dx_0$ and grid scale factor $\nu$ were taken to be 0.005 and 1.026 respectively, giving a final mesh point of 33.422. For values of $x$ exceeding this, the asymptotic estimate (12) was used.

For the heat transfer calculations the region $0 \leqq y \leqq S(x)$ was discretised using 100 equally spaced mesh points. Values of $h$ were chosen to maintain a stability factor of $\mu = 0.4$. A number of numerical tests showed that the results were insensitive to further increases in grid size.

Figure 3 shows film cooling effectiveness as a function of $x$ for a slot width of 1 and various values of $\alpha$. We note that for values of $\alpha$ close to unity the film cooling effect is minimal. As $\alpha$ decreases from 1, a greater portion of the downstream surface is protected. The exponential nature of the effect suggested by Eq. (20) is confirmed. From a practical point of view, the fact that

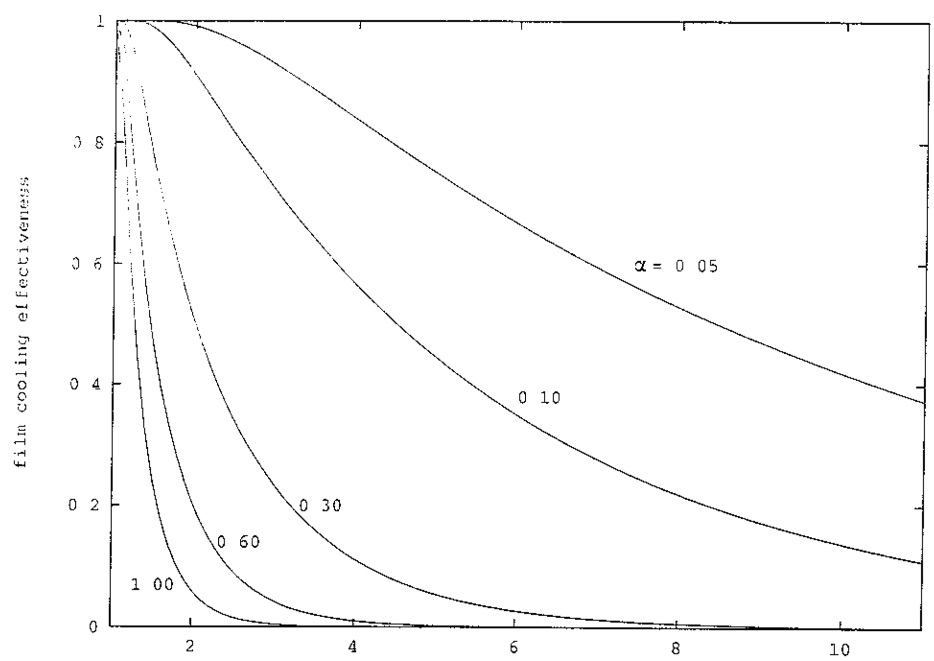


Fig. 3. Film cooling effectiveness as a function of downstream distance for various values of $\alpha$

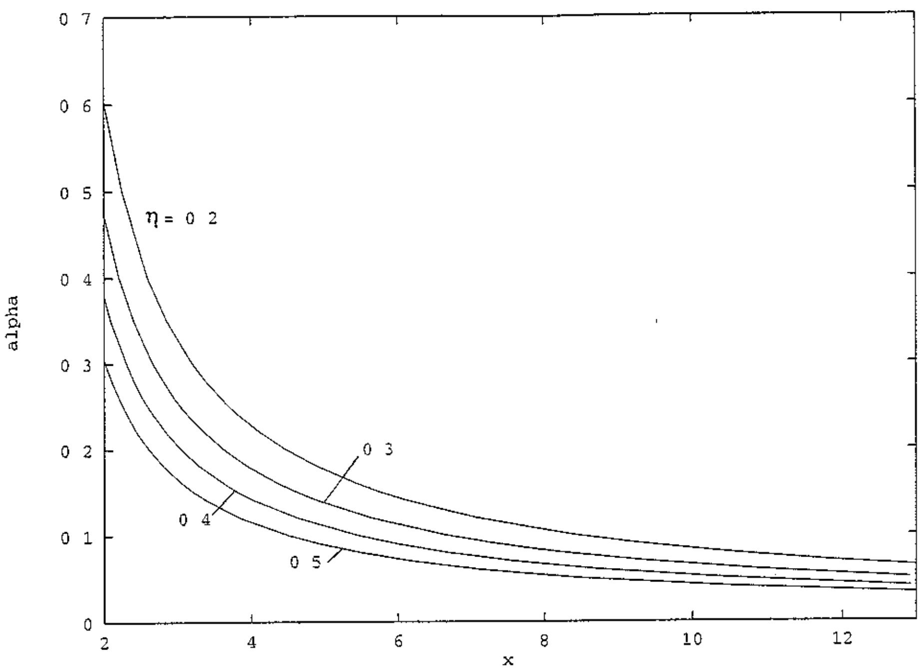


**Fig. 4.** Critical film cooling effectiveness values

the efficiency of the film cooling is determined solely by the single parameter $\alpha$ is the most important result. Since the injection velocity is given by $U_i = MU_\infty \varepsilon^3$, the "blowing rate" $B = U_i/U_\infty$ is given by $B = \varepsilon^3 M$. Defining the Prandtl number in the usual way and basing the Reynolds number Re on the slot width and $U_\infty$, we find that $\alpha$ may also be written

$$\alpha = \frac{M^{4/3}}{\Pr \mathrm{Re}\, B^{5/3}} \tag{21}$$

A doubling say of the blowing rate thus decreases $\alpha$ by a factor of about 0.315, indicating that the efficiency of the film cooling may be altered significantly by fairly modest changes in the blowing rate.

In many practical applications, design specifications dictate that the film cooling effectiveness must not fall below a certain value at a particular downstream station. Figure 4 shows the positions at which the film cooling effectiveness takes a given value as a function of $\alpha$. By using Eq. (21) the blowing rate required to satisfy specific design criteria could be determined.

Figure 5 shows comparisons between the asymptotic solution given by Eq. (20) and numerical solutions calculated using Eqs. (17) and (18). Results for the values $\alpha = 0.1$ and $\alpha = 0.05$ are shown, the numerical results being denoted by bold lines and the asymptotic estimates by broken lines. For the case $\alpha = 0.1$ the asymptotic estimate provides very acceptable predictions of film cooling effectiveness as long as $x$ exceeds approximately 7. For the smaller value of $\alpha$ the errors (as one might expect) are slightly larger.

We conclude from the results presented above that the model proposed in [2] for slot injection flow may be extended to allow the prediction of film cooling effectiveness. For reasons of brevity only simple geometries have been considered; for angled slots and slots possessing steps, ramps and other complications the flow may also be determined (see [6]). In these cases, the

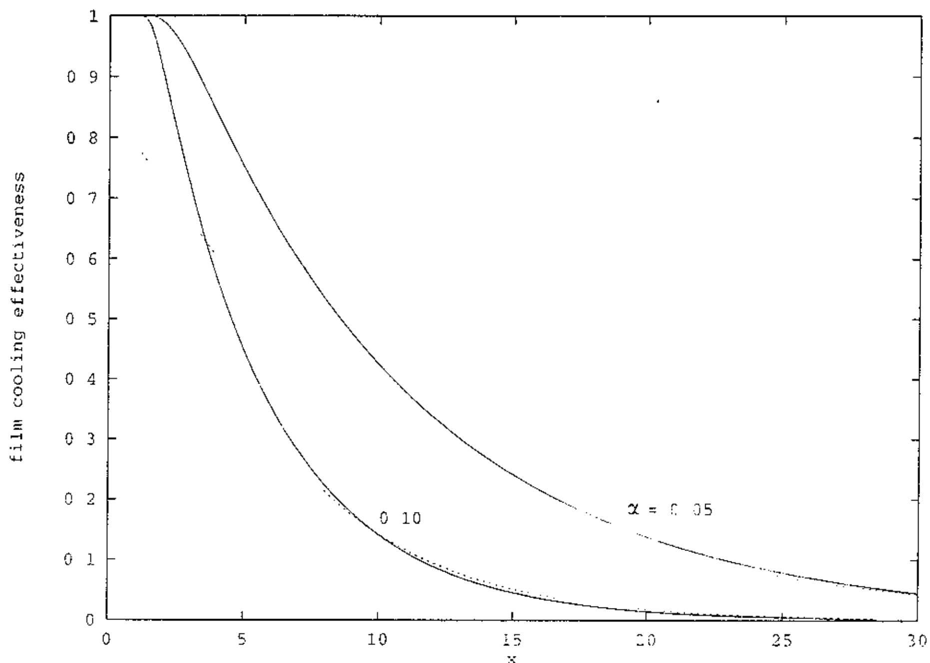


**Fig. 5.** Comparisons between asymptotic estimates and numerical calculations of film cooling effectiveness $\eta$ (Numerical solution· solid lines)

corresponding thermal problem could be solved by making some obvious changes to the model presented above. It is also possible to consider weakly compressible flows and flows where the densities of the main stream and the injected flows are constant but unequal.

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