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# HYPERBOLICITY MAPS FOR TWO-PHASE FLOW PROBLEMS

A D. Fitt
*Faculty of Mathematical Studies*
*University of Southampton*
*Southampton SO9 5NH, UK*

ABSTRACT The governing equations for gas/particulate two-phase flow with added terms are analysed Making extensive use of symbolic algebra to perform the lengthy calculations involved, parameter regimes are determined within which the conservation laws are hyperbolic and the problem is therefore well-posed

## 1. The Equations of Motion for Gas/Particulate Two-phase Flow

The precise formulation of systems of conservation laws governing multiphase flow is a topic which has long caused controversy. Put plainly, the nub of the problem is as follows : if we proceed using a traditional 'control-volume' approach to formulate the equations for inviscid flow, a method which is known to prove successful for single phase flows, then the matrix $A$ in the resulting system of first order partial differential equations

$$w_t + Aw_x = 0$$

where $w$ is the vector of unknowns, may prove to possess complex eigenvalues. This is contrary to physical intuition and renders the problem ill-posed, so that correct specification of boundary data, computation of stable numerical solutions and accurate wave speed predictions are all impossible. This defect has a simple cause; the dependent flow variables in the equations represent quantities arising from (both ensemble and cross-sectional area) averaging The averaging procedure is unavoidable if we wish to avoid the hopelessly complicated task of tracking each phase boundary, but frequently is not carried out with sufficient care, leading to the neglect of important physical terms. Drew & Wood (1985) showed how to carry out the required averaging rigourously, and emphasized the need for a careful non-dimensional analysis to decide the dominant terms in the equations of motion for any given two-phase flow regime

Concentrating in particular on gas/particulate flow where grains of reactive solid are ignited and gasified inside a tube of constant cross-sectional area, the equations of motion may be shown to be (subscripts indicating differentiation, $x$ along the tube axis)

$$(\alpha_1\rho_1)_t + (\alpha_1\rho_1 u_1)_x = \dot{m}$$

$$(\alpha_2\rho_2)_t + (\alpha_2\rho_2 u_2)_x = -\dot{m}$$

$$(\alpha_1\rho_1 u_1)_t + (\alpha_1\rho_1 u_1^2 C_{u1})_x + \alpha_1 p_{1x} = -C_s\rho_1(u_1-u_2)^2\alpha_{1x} +$$

$$(\alpha_1\phi_T\alpha_2\rho_1(u_1-u_2)^2)_x + C_{vm}\alpha_2\rho_1[(u_{1t}+u_1u_{1x}) - (u_{2t}+u_2u_{2x})] + \dot{m}u_2 +$$

$$C_D\alpha_2\rho_1(u_1-u_2)^2-\alpha_1\rho_1 g\sin\theta$$

$$(\alpha_2\rho_2u_2)_t+(\alpha_2\rho_2u_2^2C_{u2})_x+\alpha_2p_{1x}=\alpha_2[C_s\rho_1(u_1-u_2)^2]_x+$$

$$(\alpha_2^2\rho_2\phi_T(u_1-u_2)^2)_x-C_{vm}\alpha_2\rho_1[(u_{1t}+u_1u_{1x})-(u_{2t}+u_2u_{2x})]-$$

$$(\alpha_2\rho_1c_1^2\alpha_0^2\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_0}\right))_x-\dot{m}u_2-C_D\alpha_2\rho_1(u_1-u_2)^2-\alpha_2\rho_2 g\sin\theta$$

$$(\alpha_1\rho_1(e_1+u_1^2/2))_t+(\alpha_1\rho_1u_1C_{e1}(e_1+u_1^2/2))_x+(p_1u_1\alpha_1)_x+p_1\alpha_{1t}=$$

$$C_s\rho_1(u_1-u_2)^2\alpha_{1t}-(d_L\alpha_2(u_1-u_2)^3)_x+(\alpha_1u_1\phi_T\alpha_2\rho_1(u_1-u_2)^2)_x-$$

$$d_T(\alpha_2\rho_1(u_1-u_2)(e_1-T_2))_x+u_2C_{vm}\alpha_2\rho_1[(u_{1t}+u_1u_{1x})-(u_{2t}+u_2u_{2x})]+$$

$$C_D\alpha_2\rho_1u_2(u_1-u_2)^2-\alpha_1u_1\rho_1 g\sin\theta+\dot{m}(e_2+u_2^2/2)$$

Space permits only the briefest explanation of the terms in these equations; a subscript 1 refers to the gas and 2 to the solid phase, $\alpha$ represents void fraction ($\alpha_1+\alpha_2=1$), whilst $u$, $p$, $\rho$, $e$ and $T$ represent respectively velocity, pressure, density, internal energy and temperature. $C_{u1}$, $C_{u2}$ and $C_{e1}$ are profile coefficients which arise from the non-commutativity of the operations of averaging and multiplication, $\dot{m}$ characterizes the interfacial mass transfer which arises from the burning, $\alpha_0$ is the settling porosity at which elastic intergranular waves may propagate at speed $c_1$, whilst $g$ is the acceleration due to gravity and $\theta$ the inclination of the tube to the vertical. $C_s$, $\phi_T$, $C_{vm}$, $C_D$, $d_L$ and $d_T$ are coefficients of added terms, representing the effects respectively of interfacial pressure differences, turbulence, virtual mass, interphase drag, laminar interphase heat transfer and turbulent interphase heat transfer. Throughout we assume that since the solid phase is composed of incompressible particles, $\rho_2$ is constant. The system is closed by selecting the relevant constitutive equation for the gas. In this case the perfect gas law

$$e_1=\frac{p_1}{\rho_1(\gamma-1)}$$

has been used, where $\gamma$, assumed to be a known constant, is the ratio of specific heats.

## 2. Hyperbolicity Maps

Our goal is to examine the effect of different values of the added terms upon the hyperbolicity of the resulting system. In principle, this is easy to undertake; the equations are written as a $5\times5$ first order system for the unknown quantities $p_1$, $\rho_1$, $u_1$, $u_2$ and $\alpha_1$ and the eigenvalues are examined to determine whether the imaginary parts are all zero so that the system is hyperbolic. In practice however the calculations involved are very large and a symbolic algebra system (MAPLE was used in the current study) together with a custom-written package for performing the calculations (See Fitt (1987)) was employed.

When all the added terms are included the eigenvalues are given by the roots of a quintic equation containing 4759 terms, and although some analysis is possible for this case we prefer here to examine only the effect of the interfacial pressure, virtual mass and turbulence coefficients. Accordingly the values $C_{u1}=C_{u2}=C_{e1}=1$ and $c_1=d_L=d_T=C_D=0$ have been used. In this case the eigenvalues are determined by a quintic equation which contains $C_{vm}$, $C_s$ and $\phi_T$ as parameters, and depends on $R=\rho_1/\rho_2$, $V^2=(u_1-u_2)^2/c^2$, (where $c^2=\gamma p_1/\rho_1$) and $\alpha_1$. For fixed $R$ and $\gamma$ (unless otherwise stated, the physically realistic values of 1/5 and 6/5 respectively were used in all the calculations reported below) hyperbolicity regions for the equations may be examined by plotting $\alpha_1$ against $V^2$.

First we consider the case when there are no added terms so that $C_{vm} = C_s = \phi_T = 0$. For these values it may be shown that the equations are hyperbolic only if $\alpha_1 = 0$ or $V = 0$, or if $V^2 > (1 + q^{1/3})^3$ where $q = \alpha_2\rho_1/(\alpha_1\rho_2)$. Each of the four figures 1(a) - 1(d) indicates the boundary of this region (denoted L) as a solid line. In this case, just over 32% of the region $D = \{(\alpha_1, V^2) : \alpha_1 \leq 1, 0 \leq V^2 \leq 6\}$ comprises parameter space where the equations are hyperbolic. Moreover, any flows which start from rest must pass through elliptic regions of parameter space. Figure 1(a) shows the hyperbolicity map for the values $\phi_T = -1/4$ and $C_{vm} = 1/2$ (values previously suggested by diverse authors) and $C_s = -7/16$ (spherical solid particles). The hyperbolicity map is completely changed (hyperbolic regions indicated by H) but still only 38% is hyperbolic.

Some more general conclusions may be drawn by considering the effects of specific parameters; whilst no situations have been observed in which the virtual mass term helps the situation, in general the smaller the value of $\phi_T$, the greater the hyperbolic region. Figure 1(b) shows the case $C_{vm} = 0$, $C_s = 0$ and $\phi_T = -5$, when 87% of parameter space is hyperbolic. An even better result may be obtained by the choice $C_{vm} = \phi_T = 0$, $C_s = -1$, when it is possible to show that the equations are hyperbolic everywhere. Figure 1(c), where $C_s = -99/100$ and the other coefficients are zero, shows that this is a somewhat singular limit however; here only 73% of parameter space is hyperbolic.

Finally, the influence of the density ratio $R$ may be examined. Figure 1(d) shows a hyperbolicity map for the case $C_{vm} = 0$, $C_s = -7/16$ and $\phi_T = -1$ so that the particles are spherical, virtual mass effects are ignored and the turbulent effects are fairly strong. The density ratio in this case is $R = 1/100$ (heavy particles) and 56% of parameter space is hyperbolic. For the same values of $C_{vm}$, $C_s$ and $\phi_T$, but a density ratio of 1/5, 68% of parameter space is hyperbolic.

## 3. Conclusions

Two phase flows are undoubtedly highly complicated phenomena and any attempt to construct mathematical models must reflect this complexity. The diversity of possible flow regimes and qualitative behaviours makes it unrealistic to expect simple models to be successful, and this manifests itself in the non - hyperbolicity of the governing equations. The work outlined above provides a process by which 'incorrect' models may be identified and discarded. However, even if submodels and correlations are used which render the equations totally hyperbolic, the hyperbolicity in itself does not *prove* that the equations are necessarily correct. As in all highly involved modelling problems, careful dimensional analysis, comparison with experiment and shrewd physical intuition are obligatory requirements for successful predictions.

## References

Drew, D.A. & Wood, R.T. (1985) 'Overview and Taxonomy of Models and Methods for Workshop on Two-phase Flow Fundamentals' National Bureau of Standards Report, Gaithersburg, Maryland

Fitt, A.D. (1987) 'Symbolic Computation of Hyperbolicity Regions for Systems of Two-Phase Flow Conservation Laws Using Maple', J. Symbolic Computation 8 305-308

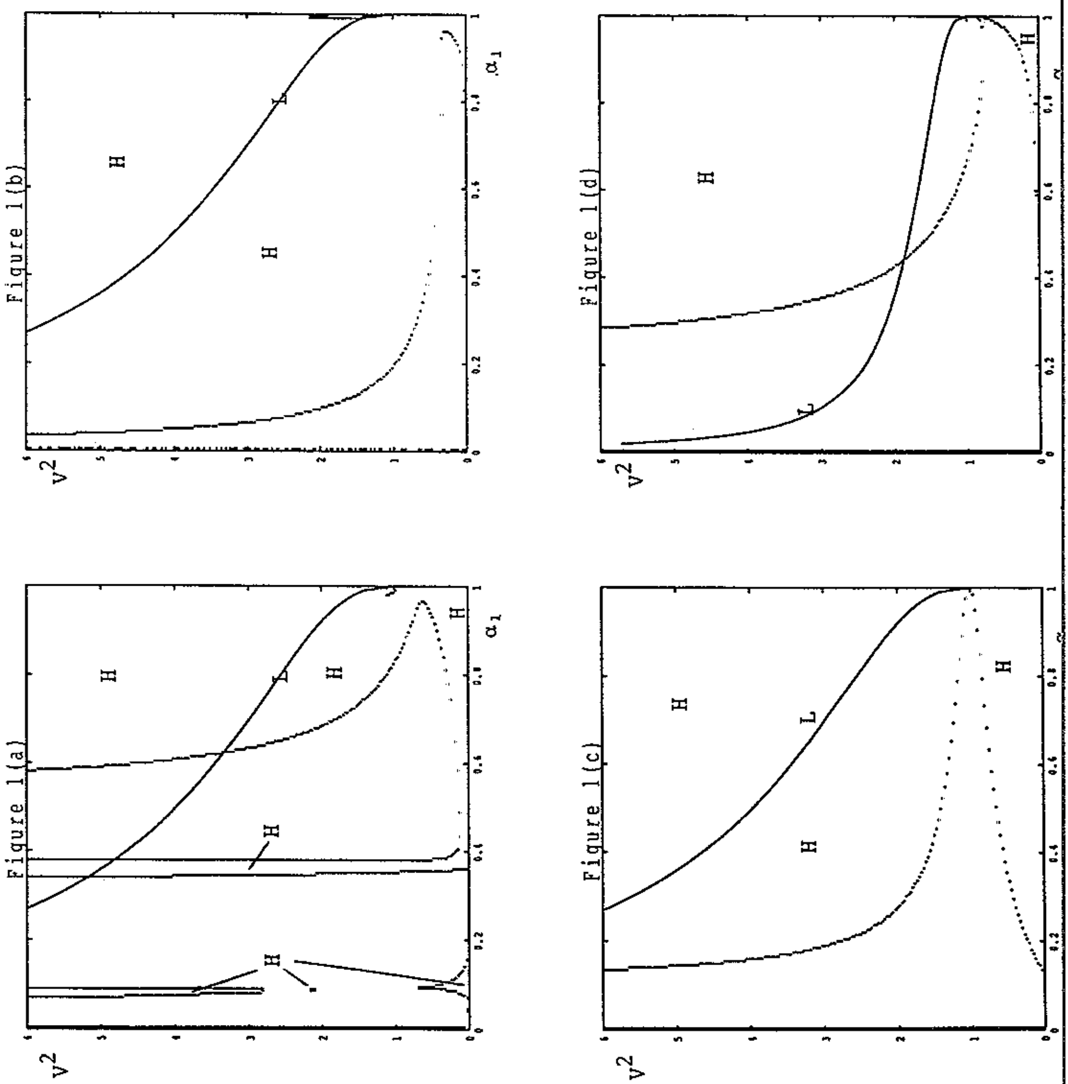
Figure 1(a)
Figure 1(b)
Figure 1(c)
Figure 1(d)
$V^2$
$\alpha_1$
H
L