A SLOW FLOW PROBLEM ARISING IN CONTINUOUS ELECTRODE SMELTING

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ABSTRACT A slow flow model is proposed for the determination of the effective viscosity of a sample of pitch and anthracite electrode material. The case of a tall, thin sample is considered and a simple formula is derived for the viscosity. The boundary layer structure at the edges of the sample is also considered. Finally, numerical calculations are performed using a boundary element method

1. Continuous Electrode Smelting

The problem considered herein concerns the determination of the 'effective viscosity' of a material used to make continuously consumed electrodes, whose function is to conduct large amounts of electrical energy to the centre of a blast furnace for the production of various kinds of alloy and steel. The material, traditionally known as 'paste', is composed of a mixture of anthracite fines of widely differing sizes, held together by a pitch binder. The mixture is solid at room temperature but begins to flow as the temperature is increased. Although the mixture is clearly a multiphase fluid, and under some circumstances can exhibit segregation, or phase separation (for details of a theoretical study examining this phenomenon see BERGSTROM et al. (1989)) there is much practical interest in determining its effective viscosity by treating it as though it were a single phase mixture. By further assuming that the paste behaves like a Newtonian (highly) viscous fluid, we acknowledge the fact that although this is certainly a large oversimplification, an accurate constitutive law for the mixture would not be easy to propose because of the wide variation in size of the anthracite fines, and sensitive dependence on temperature.

For the purposes of this study we will assume that the temperature is fixed and consider one of the many different tests used, known as the 'velocity test.' Here a sample of the paste is placed on a flat surface, and a moveable plate is placed on the top of the sample. The plate is moved vertically downwards with a prescribed velocity, squashing the sample, and the effective viscosity is inferred from the 'bulge' exhibited by the lower portion of the sample. (There are other tests including those involving prescribed loads which may be treated similarly, but space does not permit their discussion here. [1])

[1] Details of other tests may be found in FITT & AITCHISON (1991).

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2. A Slow Flow Model

To model the flow of the paste sample, we begin by non-dimensionalizing the Navier-Stokes equations. In order to simplify the analysis presented here, we assume that the sample is a two-dimensional rectangle; normally in practice the sample is cylindrical, and such geometries can be dealt with in a exactly similar fashion to that described here Taking unit vectors $\boldsymbol{i}, \boldsymbol{j}$, and assuming that the sample has height h and semi-width L, we scale lengths with h, velocities with a representative velocity U_∞, time with h/U_∞ and pressures and stresses with $\mu U_\infty/h$. This gives the equations

$$Re[\boldsymbol{q}_t + (\boldsymbol{q}\ \nabla)\boldsymbol{q}] = -\nabla p + \nabla^2 \boldsymbol{q} - \frac{Re}{Fr}\boldsymbol{j}, \qquad \nabla . \boldsymbol{q} = 0$$

Here the Reynolds and Froude numbers are defined respectively by $Re = hU_\infty \rho/\mu$, $Fr = U_\infty^2/gh$ With typical values of $h \sim 1\text{m}$, $U_\infty \sim 1\text{m/hr}$, $\rho \sim 3\text{gm/cm}^3$ and $\mu \sim 10^8$ Pa sec, we find that $Re \sim Fr \sim O(10^{-8})$, leaving the slow flow equations with body force·

$$\nabla p = \nabla^2 \boldsymbol{q} - \alpha \boldsymbol{j}, \qquad \nabla . \boldsymbol{q} = 0$$

We impose standard no-slip boundary conditions on the bottom surface $y = 0$, stress-free conditions and the usual kinematic constraint on the sample side $x = \xi(y,t)$ say, whilst on the top surface we have $u = 0$, $v = \dot{s}(t)$, corresponding to the imposed velocity of the 'pusher plate' To attack the full problem posed here a numerical approach is required However, there are some situations in which analytical results may be obtained.

3. A Tall, thin sample.

Suppose the sample is tall and thin so that $L/h = \epsilon \ll 1$ Making the scalings $x = \epsilon X$, $u = \epsilon U$ and $\xi = \epsilon\eta$, the equations of motion become

$$p_X = U_{XX} + \epsilon^2 U_{yy}, \quad \epsilon^2 p_y = v_{XX} + \epsilon^2 v_{yy} - \epsilon^2\alpha, \quad U_X + v_y = 0$$

These may easily be solved by setting $U = U_0 + \epsilon U_1 + \epsilon^2 U_2 + \dots$ and using similar expansions for v, p and η. Imposing symmetry conditions $U = v_X = 0$ on $X = 0$, the kinematic boundary condition on the free surface, and considering the y-component of the stress-free boundary condition on the free surface to order ϵ^2, we find that the solution is

$$v = v_0(y,t) + \epsilon v_1(y,t) + \epsilon^2[\frac{X^2}{2}(\alpha - 3v_{0yy}) + q(y,t)]$$

$$U = -Xv_{0y} - \epsilon X v_{1y} + \epsilon^2[\frac{X^3}{2}v_{0yyy} - Xq_y]$$

where η_0 and v_0 satisfy

$$\dot{\eta}_0 + (\eta_0 v_0)_y = 0, \quad 4(\eta_0 v_{0y})_y = \alpha\eta_0$$

These equations admit two boundary conditions for v_0 and an initial condition for η_0 Clearly the correct ones to impose are $v_0 = 0$ on $y = 0$ and $v_0 = \dot{s}(t)$ on $y = s(t)$; evidently the boundary conditions for u cannot be satisfied and there will be 'inner' regions near to the top and bottom of the sample where a boundary layer analysis will be required to complete the details of the solution. The two equations for η_0 and v_0 are in general not easy to solve, but for small times the viscosity may be estimated from the bulge in the following manner. Assuming that $\eta_0(y,0) = 1$ and $v = -V_T$ on $y = 1$ where V_T is non-dimensional velocity, we find that

$$\eta_0 = 1 + \frac{t}{8}(8V_T - 2\alpha y + \alpha) + O(t^2)$$

$$v_0 = \frac{\alpha y^2}{8} - y\left(V_T + \frac{\alpha}{8}\right) + \frac{t}{192}(\alpha y(2\alpha y^2 - 3\alpha y + \alpha - 24yV_T + 24V_T)) + O(t^2)$$

It has already been noted that this solution is not valid near to $y = 0$ We shall assume however that the maximum 'bulge' is occurs at the boundary layer edge, which corresponds to the point $y = 0$ Re-dimensionalizing shows that if the top plate moves with velocity U_∞ and the maximum semi-width of the sample is BL, then this is related to the effective viscosity by

$$\mu = \frac{h^2 \rho g t}{8((h(B-1) - U_\infty t)}$$

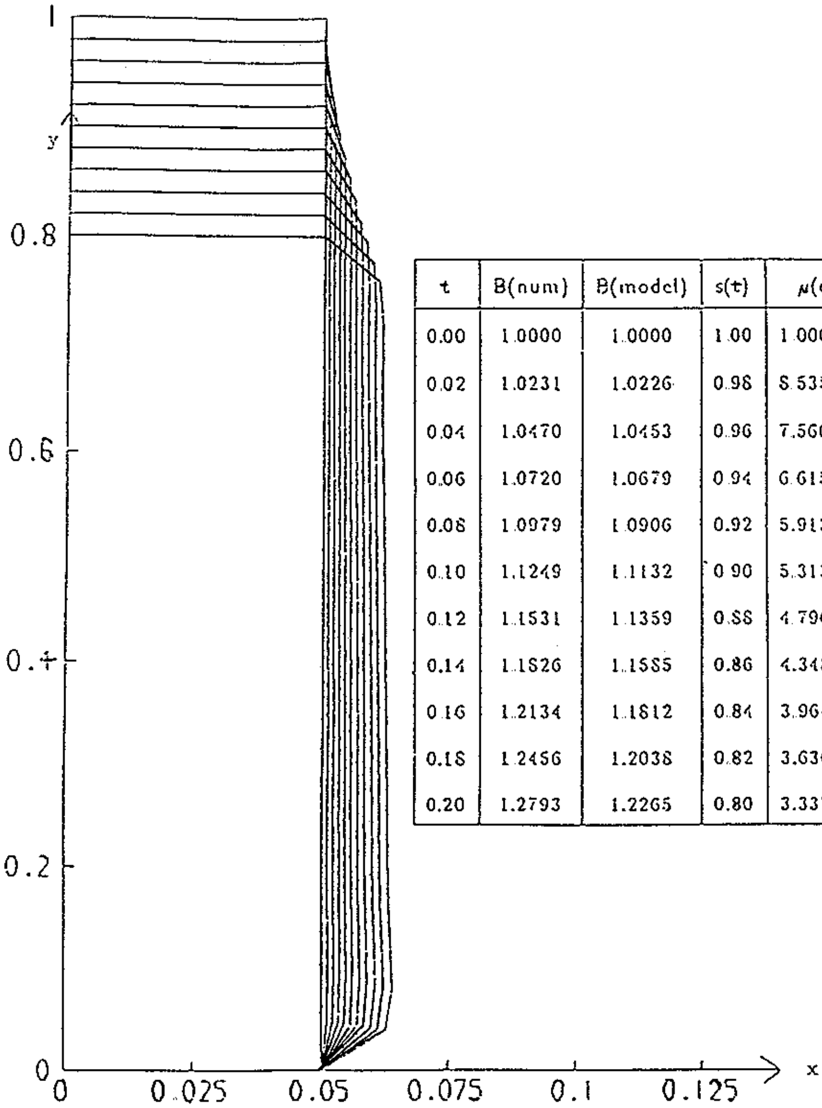
Unfortunately experimental results are unobtainable, so the best which can be achieved is to compare the theory with a numerical 'experiment'. Figure (1) shows a comparison between the theory and a numerical solution to the full problem which was calculated by a finite element method in which the (p,u,v)-version of the Stokes equations was written in stress-divergence form so that the Galerkin method gave the stress conditions as natural boundary conditions The topology of the grid was maintained by the time-stepping of exterior and interior nodes. The aspect ratio of the 'experimental' sample was 10·1 and the 'experimental' viscosity was 10^8 Pa Sec. Clearly for small times the theory gives a satisfactory estimate of the viscosity

Some consideration of the boundary layers which exist near to the top and bottom of the sample is necessary. Considering the bottom of the sample (the top may be treated similarly), it is clear that the additional scalings $y = \epsilon Y$, $v = \epsilon V$ must be made This leads to the equations

$$P_X = U_{XX} + U_{YY}, \quad p_Y = V_{XX} + V_{YY} - \epsilon\alpha, \quad U_X + V_Y = 0$$

with boundary conditions $U = V = 0$ on $Y = 0$, stress-free conditions on $X = \eta(Y,t)$, and symmetry on $X = 0$ There is also a matching condition which must be satisfied as $Y \to \infty$ For small times the matching condition is known from the solution given above, the free boundary remains vertical to lowest order, and a stream function Ψ may be introduced which satisfies the biharmonic equation. By setting $\Psi = -KYX + \Phi$, where $K = -(V_T + \alpha/8)$, in order to retrieve a problem where $\Phi \to 0$ as $Y \to \infty$, we are finally required to find a function $\Phi(X,Y)$ which is biharmonic on the semi-infinite strip $\{Y \geq 0, X \in [0,1]\}$, vanishes as $Y \to \infty$, and satisfies $\Phi = \Phi_{XX} = 0$ at $X = 0$, $\Phi = 0$, $\Phi_Y = KX$ at $Y = 0$ and $\Phi_{YY} - \Phi_{XX} = \Phi_{XXX} + 3\Phi_{YYX} = 0$ at $X = 1$. This may be accomplished fairly easily by writing Φ as a Papkovitch-Fadle eigenfunction expansion (for fuller details see, for example SPENCE (1978)) in the form

$$\Phi(X,Y) = \sum_n c_n \phi_n(X) e^{-\lambda_n Y}$$



t	B(num)	B(model)	s(t)	μ(expt)
0.00	1.0000	1.0000	1.00	1.000×10^8
0.02	1.0231	1.0226	0.98	8.535×10^8
0.04	1.0470	1.0453	0.96	7.560×10^8
0.06	1.0720	1.0679	0.94	6.615×10^8
0.08	1.0979	1.0906	0.92	5.913×10^8
0.10	1.1249	1.1132	0.90	5.313×10^8
0.12	1.1531	1.1359	0.88	4.796×10^8
0.14	1.1826	1.1585	0.86	4.348×10^8
0.16	1.2134	1.1812	0.84	3.964×10^8
0.18	1.2456	1.2038	0.82	3.630×10^8
0.20	1.2793	1.2265	0.80	3.337×10^8

Figure 1 : Sample shape, numerical calculations and comparison with theory

where

$$\phi_n(X) = \left(\frac{1}{\lambda_n \cos^2 \lambda_n}\right)(X \cos\lambda_n \cos\lambda_n X + \sin\lambda_n \sin\lambda_n X)$$

(the scaling factor has been included for convenience) and the Papkovitch-Fadle eigenvalues are members of the doubly-infinite family complex solutions to

$$\lambda + \sin\lambda\cos\lambda = 0$$

with real part greater than zero. The relevant eigenvalues may easily be computed using the Newton-Raphson method, for example, and to complete the details of the boundary layer structure it remains only to show that the eigenfunction coefficients c_n may be determined to satisfy the boundary conditions

$$0 = \sum_n c_n\phi_n(X), \qquad KX = \sum_n -\lambda_n c_n \phi_n(X).$$

Ostensibly the task of finding complex numbers c_n to satisfy the above conditions is easy; matters are complicated however by the fact that the eigenfunctions are not mutually orthogonal. In some circumstances this defect may be remedied by constructing so-called biorthogonal functions. To be specific, suppose we define

$$\boldsymbol{f} = (f_1(X), f_2(X), f_3(X), f_4(X)) = (\Phi_{XY}, \Phi_{XX}, Q, P)\,|_{y=0}$$

where $P = \nabla^2\Phi$ and Q is the harmonic conjugate of P. If we now use the eigenfunction expansion to define functions $\phi_{nk}(k = 1, 4)$ via the expression

$$f_k(X) = \sum_n c_n\phi_{nk}(X),$$

Then it is possible to find 'biorthogonal' functions $\beta_{m1}(X)$ and $\beta_{m3}(X)$ so that

$$\int_0^1 \beta_{m1}\phi_{n1} + \beta_{m3}\phi_{n3}\,dX = \delta_{mn},$$

so that the coefficients c_n may be calculated directly by quadrature. The same procedure is possible for the functions ϕ_{n2} and ϕ_{n4}, so that when the problem is such that either the f_1 and f_3 or the pair f_2 and f_4 are prescribed as data, calculation of the coefficients is easy. Unfortunately only these two data prescriptions may be dealt with in this way - for other 'non canonical' problems such biorthogonal functions do not exist. Since in our case the data prescribed was Φ and Φ_y, which amounts to f_1 and f_2, the problem is of non-canonical type.

One obvious method calculating the c_n is by collocation, which gives rise to an infinite set of linear equations. It is then possible to proceed by solving a truncated (say $N \times N$) system of equations to approximate the c_n. During this procedure, great care must be taken to ensure that a diagonally dominant matrix is produced, thereby ensuring that as $N \to \infty$ the solutions to the truncated system converge to the solutions of the infinite system. This may however be accomplished by using the 'optimal weighting functions' introduced by SPENCE (1978), so completing the determination of the boundary layer structure.

4. Conclusions

A model has been presented which allows the effective viscosity of materials whose flow is governed by the slow flow equations to be determined. In the case of a tall, thin sample simple estimates for the viscosity may be found which give acceptable agreement with numerical calculations for the full problem performed using the boundary integral method. The boundary layer structure at the top and bottom of the sample leads to a non-canonical biharmonic problem which is of some interest in itself For the problem with more general geometries however, numerical methods must be used.

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