The closed-form integration of arbitrary functions

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Introduction

Consider the usual experience of a student who progresses far enough in school mathematics to begin to study the calculus. After motivation of the topic of "rates of change", simple differentiation of polynomials is learned. More advanced functions are then considered, and eventually the student meets the product, quotient and chain rules. The result: with enough algebraic accuracy and persistence, the student can determine derived functions for virtually any sufficiently well-behaved combination of standard functions. A natural continuation of calculus is the determination of areas under curves and volumes of revolution, which entails discussion of the fundamental theorem of calculus. This asserts that the "area function" $\mathrm{g}(x)$ defined by

$$\mathrm{g}(x) = \int^{x} \mathrm{f}(t)\,\mathrm{d}t$$

has derivative $\mathrm{f}(x)$. This leads to the problem of finding "anti-derivatives" – given a function $\mathrm{f}(x)$, can we determine another function $\mathrm{g}(x)$ such that $\mathrm{g}'(x) = \mathrm{f}(x)$? Again, some standard examples are usually considered first, before integration by substitution, by parts and other methods are examined. Indeed, the beginnings of an algorithmic approach are considered, based essentially on Hermite's result that the integral of a rational function of one real variable is elementary (see below for definition) since it is a linear combination of logarithms, inverse tangents and rational functions.

Consider now the student's plight! Does he or she not have a right to expect that since (i) the link between differentiation and integration has been emphasized throughout and (ii) he or she can calculate "any" derivative, it should be possible to perform any anti-differentiation? Evidence that this is not the case normally involves such functions as

$$\mathrm{e}^{-x^2}, \ \frac{\sin x}{x} \ \text{ or } \sqrt{\sin x}$$

and the message to the student, once understood, is the same: although the problem of differentiation is formally solved, finding an anti-derivative for a given function at best involves locating the correct technique in a somewhat arbitrary manner, and at worst complete failure. The fact that similar integrands give wildly different results (consider for example $(1+x)^{-2}$, $(1+x)^{-1}$, $x(1+x^2)^{-1}$, $(1+x^2)^{-2}$, and $(1-x^2)^{-1}$) serves only to add to the general confusion.

Bearing this in mind, is there any way of tackling the integration problem, other than by using a combination of random techniques, experience and luck? The purpose of this article is to point out that not only are there better ways of proceeding, (some of which are of great practical use and are employed in the new generation of symbolic algebra packages) but also that, in some senses, the integration problem may be regarded as completely solved. It is surprising how little known this result seems to be. Even more surprising is the fact that the theory was initially established by Liouville (who was in turn motivated by some conjectures of Abel) more than 150 years ago, and, for the most part, does not require a great degree of mathematical sophistication. It should be stressed that in all of the discussion below, we are concerned only with the *formal* problem of finding antiderivatives, rather than foundational notions such as the Lebesgue integral or measure theory.

Some definitions

We assume throughout that we are interested in the purest form of the integration problem: given $f(x)$ can we determine an antiderivative $g(x)$ such that $g'(x) = f(x)$? As usual, we do not expect $g(x)$ to be determined uniquely, as an arbitrary constant may always be added. The additive constant is irrelevant for the purposes of the discussion below, however, and is taken to be zero throughout. Of course we have to decide what we regard as "allowable candidates" for a solution. In this article we shall restrict our attention to **elementary** functions. By elementary functions we mean those built up from rational functions of x by successively exponentiating, taking logarithms, and performing algebraic operations (that is, solving polynomial equations whose coefficients are previously defined functions). We also choose to use complex coefficients throughout, since the set of elementary functions then includes sines, cosines and their inverses, by using, for example, Euler's formula

$$\cos x = (e^{ix} + e^{-ix})/2.$$

For our purposes, this definition of the elementary functions will prove sufficient, and the reader is encouraged to equate the concept "elementary function" with "functions that may be built up with a scientific calculator using a finite number of operations". However for total rigour more care is required in the framing of the definition of elementary function and full details are given in [1].

It is also important to realize that the point at issue here does not concern "whether an answer exists", but whether it exists *in a particular form*. An analogy may be drawn with the solution of polynomial equations: given a quadratic equation we could either determine the roots by using the formula, or employ a numerical method such as Newton-

Raphson. In the spirit of the present discussion we are interested only in the former method of solution, that is, *is there a formula*? For general polynomials it was proved by Abel that a general formula exists only when the polynomial has degree four or less, but what can be said about the integration problem? Some feel for the sort of results that can be established may be gained from the result that

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

for $n \neq -1$. What can be said in the special case $n = -1$? Is it possible that the answer could still be a rational function? That this cannot be the case may be easily shown by contradiction, for if it were true that

$$\int \frac{1}{x} \, dx = \frac{P(x)}{Q(x)} \qquad (1)$$

where P and Q are coprime then by differentiation

$$\frac{1}{x} = \frac{QP' - PQ'}{Q^2}$$

This gives

$$Q^2 = x(QP' - PQ')$$

and thus Q has a zero at $x = 0$. Assuming that $Q = x^n R(x)$ where $R(0) \neq 0$ and dividing by x^n we have

$$x^n R^2 = xRP' - nPR - xPR'$$

but this implies that the term PR is zero at $x = 0$, contradicting the assumption that neither P nor R has a zero at $x = 0$. Thus the integral of $1/x$ cannot have the form given by (1).

The theorem of Liouville

Liouville's most important theorem on the problem of integration concerns the form that a primitive must take if it is to be elementary. Davenport *et al* [2] state the theorem as follows:

Theorem (Liouville, 1833)[4] Let f be a function from some function field K. If f has an elementary integral over K, it has an integral of the form

$$\int f = v_0 + \sum_{i=1}^{n} c_i \log(v_i) ,$$

where v_0 belongs to K, the v_i belong to K' an extension of K by a finite number of constants algebraic over K, and the c_i belong to K' and are constant.

There may be terms in this statement with which some readers are unfamiliar, but the technical details of the theorem are not required for what follows. The theorem essentially states that if f has an elementary integral, then (by differentiation) f must be of the form

$$f = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}.$$

For the context we have in mind here, the function field K is simply the field of the elementary functions.

The theorem is one of great power, but the proof (see for example [4]) which is based on induction requires only a minimal amount of specialist knowledge. We content ourselves here with observing that if, for example, f is algebraic, (any function constructed in a finite number of steps from the operations of addition, subtraction, multiplication and division, the extraction of integral roots and from the inverses of any functions already constructed) then $\int f$ cannot include exponentials since, roughly speaking, exponentials survive differentiation. The same applies to any logarithmic term involved in $\int f$ unless it enters the expression in a linear way. Also the v_i may be shown to be algebraic since logarithms of elementary functions also "partially" survive.

Functions without elementary primitives

In what follows, we exploit a special case of Liouville's theorem to obtain a stronger version of the general result that is applicable to some special cases. Specifically it is asserted that:

Theorem ("Rational Liouville theorem") Let f and g be algebraic functions of x with g non-constant. Then if the integral of $f.e^g$ is elementary, it is given by

$$\int f.e^g\, dx = R e^g$$

where R is rational in f, g and x.

For the sake of completeness, we give a proof of this result adapted from [1, p 47]. This proof is not difficult, but a knowledge of partial differentiation is required and the reader may wish to omit it at the first reading. The key section of the proof makes use of the fact that in an identity between two functions, the independent variable may be replaced by some other variable (including one that contains a parameter) and the expression may be differentiated or integrated. As an example of this, consider the identity

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Replace θ by $\mu\theta$ and differentiate with respect to μ. This gives

$$2\theta\cos 2\mu\theta = 2\theta\cos^2\mu\theta - 2\theta\sin^2\mu\theta.$$

Now letting $\mu = 1$ yields the other "double angle" formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Some readers will have encountered similar manipulations when calculating the envelopes of families of curves. The only other result that we ask the reader to assume is that the exponential of a non-constant algebraic function cannot itself be an algebraic function, and is henceforth referred to as "transcendental".

Proof of the Rational Liouville theorem Assume that f and g satisfy the conditions of the theorem, and that the integral of fe^g is denoted by u. By Liouville's theorem, if u is elementary then it must have the form

$$u = v_0 + \sum_{i=1}^{n} c_i \log(v_i) .$$

Moreover, denoting $e^{g(x)}$ by θ, this may be written

$$u = v_0(\theta, x) + \sum_{i=1}^{n} c_i \log(v_i(\theta, x)) , \tag{2}$$

with each of the v_i rational in θ, x, f and g. The equation (2) is an identity in the variable x, but we now regard θ and x as independent variables. Differentiating with respect to x and using the chain rule gives

$$\frac{du}{dx} = f\theta = \theta g'(x)\frac{\partial v_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + \sum_{i=1}^{n} \frac{c_i}{v_i}\left(\theta g'(x)\frac{\partial v_i}{\partial \theta} + \frac{\partial v_i}{\partial x}\right) . \tag{3}$$

Our goal is now to derive a partial differential equation for $u(\theta, x)$, and the fact that θ is transcendental guarantees that the identity (3) is an identity in both x and θ, and it is therefore permissible to replace θ with $\mu\theta$ whilst leaving x unchanged, even though θ is itself a function of x. (See further remarks below.) An integration with respect to x now gives

$$\mu u = v_0(\mu\theta, x) + \sum_{i=1}^{n} c_i \log(v_i(\mu\theta, x)) + C(\mu) \tag{4}$$

where $C(\mu)$ is an arbitrary "constant". From (2) we see that

$$u(\mu\theta, x) = v_0(\mu\theta, x) + \sum_{i=1}^{n} c_i \log(v_i(\mu\theta, x))$$

and so, comparing this with (4) we find that

$$\mu u(\theta, x) = u(\mu\theta, x) + C(\mu).$$

Finally, differentiation with respect to μ gives, on setting $\mu = 1$,

$$\theta \frac{\partial u(\theta, x)}{\partial \theta} = u(\theta, x) + D$$

where $D = C'(1)$ is a constant. This may be solved as usual to yield

$$u = -D + A(x)\theta$$

and, setting $\theta = \theta_0$, we find that $A(x)$ is determined by

$$u(\theta_0, x) = A(x)\theta_0 + D$$

Thus

$$u = \theta\left(\frac{u(\theta_0, x) - D}{\theta_0}\right) + D$$

and is, indeed of the form stated.

The key step in the proof concerns the operation of replacing θ by $\mu\theta$ whilst leaving x unchanged. To understand why θ must be transcendental to allow this, consider first the case where $f = g = x$. Now (2) becomes $u = x\theta - \theta$ whilst (3) becomes $f\theta = \theta(x-1) + \theta$. Clearly if θ is now replaced by $\mu\theta$ the identity still holds. Consider however what could happen if θ were *not* transcendental. Suppose for example we tried to repeat the same argument for the integral of $f\theta$ taking $f = x$ and $\theta = x^2$. Then (2) becomes $u = x^4/4$ and the equivalent of (3) is now $f\theta = \theta + x(2x)$. A falsehood now results if the variable θ is replaced by $\mu\theta$ whilst x is left unchanged; essentially this comes about because θ and x may now be expressed in terms of each other in a non-trivial way.

The significance of the Rational Liouville theorem is that it may be used to show very quickly that certain integrands do not possess elementary primitives. For example, if the integral of e^{-x^2} was elementary, then by the theorem the primitive would have to take the form

$$\int e^{-x^2}\,dx = R(x)\,e^{-x^2}. \qquad (5)$$

Suppose that we now set $R(x) = P(x)/Q(x)$ where both P and Q are co-prime polynomials with Q non-zero. Then, if (5) is to be true then differentiation shows that

$$e^{-x^2} = \frac{QP' - PQ'}{Q^2}\,e^{-x^2} - 2x\frac{P}{Q}\,e^{-x^2}$$

and thus

$$Q(Q - P' + 2xP) = -PQ'. \qquad (6)$$

If P and Q possess zeros then they cannot be shared, as P and Q are coprime. Further, if Q possesses a (possibly complex) zero of order n at $x = a$ then Q' possesses a zero of order $n-1$. This would mean however that the left hand side of (6) has a zero of order at least n, whilst the right -hand side has a zero of order $n-1$, a contradiction. Q is therefore a polynomial with no zeros and in consequence must be constant, Q_0 say. Thus (6) yields

$$P' - 2xP = Q_0 \qquad (7)$$

Finally observe that (7) cannot be satisfied by a polynomial P, for if the degree of P is m then the degree of P' is $m-1$, whilst the degree of $2xP$ is $m+1$. But the difference between two such polynomials can never be constant, and the proof is complete.

Now that the basic method has been established, other results are easy to obtain. For example, suppose that the integral of e^x/x was elementary. Then, proceeding along similar lines to the above argument, it would be true that

$$\int \frac{e^x}{x}\,dx = \frac{P(x)}{Q(x)}\,e^x .$$

Differentiation and rearrangement gives

$$Q(Q - xP' - Px) = -xPQ'$$

and it is again fruitful to consider the zeros (if any) of $Q(x)$. First suppose that 0 is not a zero of Q. Then by the same argument as above we must have $Q = Q_0$ = a constant, and we are left to determine a polynomial P satisfying

$$xP' + xP = Q_0 .$$

Again, by considering the degree of both sides this equation may be seen to be impossible. The only remaining possibility is that 0 is the only zero of $Q(x)$, in which case it must be that $Q(x) = Kx^n$ for some constant K and $n \geqslant 1$. Then

$$Kx^n(Kx^n - xP' - Px) = -xPnKx^{n-1}$$

or

$$(Kx^n - xP' - Px) = -Pn,$$

an immediate contradiction since the left-hand side has 0 as a zero, but the right-hand side does not.

The result that e^x/x has no elementary primitive may be used to obtain other conclusions. For example, a substitution $x = \log u$ shows that the integral of $1/\log u$ cannot be elementary. More generally, a substitution $x = k \log u$ shows that

$$\int \frac{u^{k-1}}{\log u}\,du$$

is not elementary for any non-zero k. In the exceptional case $k = 0$ there is an elementary primitive of course, namely $\log\log u$. Furthermore, exploiting the fact that we are working over the field of complex numbers, it is a simple matter to show by an examination of the function e^{ix}/x, that the integral of $\sin x/x$ is not elementary. For details see [5].

As well as providing simple proofs that certain well-known functions have no elementary primitives, arguments similar to that above can be employed to investigate more complicated integrals. As an example, we determine under what (if any) circumstances we may find an elementary expression for

$$I = \int \frac{e^{-x^2}(\alpha x^3 + \beta x^2 + \gamma x + \delta)\,dx}{(x+1)^2}$$

where α, β, γ and δ are constants, not all zero. Arguing as before that the integrand, if elementary, must be of the form

$$\frac{P(x)}{Q(x)}\, e^{-x^2}$$

where P and Q are polynomials, we find by differentiation that, if this is so, then

$$Q[Q(\alpha x^3+\beta x^2+\gamma x+\delta)+2x(x+1)^2 P-(x+1)^2 P'] = -(x+1)^2 PQ'. \quad (8)$$

Assuming as usual that P and Q have no common factors, consideration of the degrees of the left and right sides of the equation at the zeros of Q shows that either (i) $Q = Q_0$, a constant, or (ii) $Q = A(x+1)^n$ for some $n > 0$. We consider case (i) first. If it is true that

$$Q_0(\alpha x^3+\beta x^2+\gamma x+\delta)+2x(x+1)^2 P-(x+1)^2 P' = 0 ,$$

then unless P has degree 0 so that $P = P_0$, there will be uncancelled terms of degree 4 or greater. So setting $P = P_0$ and collecting terms gives

$$x^3[\alpha Q_0+2P_0] + x^2[\beta Q_0+4P_0] + x[\gamma Q_0+2P_0] + [\delta Q_0] = 0.$$

If $Q_0 = 0$ then $P_0 = 0$ and the problem is the trivial one, so it must be that $\delta = 0$, in which case $\alpha = \gamma$ and $\beta = 2\gamma$ and the integral is given by

$$I = \frac{P_0}{Q_0}\, e^{-x^2} = \frac{-\gamma}{2}\, e^{-x^2} .$$

Case (ii) provides some less obvious results, for after setting $Q = A(x+1)^n$ in (8) and cancelling common factors, we find that

$$A(x+1)^{n-1}(\alpha x^3+\beta x^2+\gamma x+\delta)+2x(x+1)P-(x+1)P'+Pn = 0 .$$

Now n is an integer greater than zero, but if $n \geqslant 2$ then setting $x = -1$ in the above expression implies that P has a zero at $x = -1$, an impossibility since P and Q share no factors. Thus $n = 1$ and P is linear, say of the form $P_0 + P_1 x$. Finally equating coefficients of x gives the equations

$$\begin{aligned} A\alpha + 2P_1 &= 0 \\ A\beta + 2P_0 + 2P_1 &= 0 \\ A\gamma + 2P_0 &= 0 \\ A\delta - P_1 + P_0 &= 0 \end{aligned}$$

which may easily be solved to yield further conditions under which I is elementary, namely that $\beta = \alpha + \gamma$ and $2\delta = \gamma - \alpha$. If these pertain, then the integral is given by

$$I = \frac{-\alpha x-\gamma}{2(x+1)}\, e^{-x^2} ,$$

so completing the task of specifying precisely the conditions under which I is elementary.

Extensions to Liouville's theorem and implications for computer algebra

Having seen how Liouville's theorem allows many of the more familiar results concerning closed-form integration to be proved, the question naturally arises as to whether the theorem may be extended. In particular, it would be attractive to augment the set of "elementary" functions by adding some of the more useful special functions such as the error function, Bessel functions and so on. In general, it turns out that this is not possible (the details are somewhat involved, but this area is one of constant evolution and the interested reader is referred to the interesting discussion in [2]) and we have to be content with the class of elementary functions defined above. It is interesting to observe in passing however that another consequence of the theorems given above is that "special" functions such as erf(x) defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

really are "new" functions – however hard we try it is not possible to express them as elementary functions.

Another generalization of Liouville's theorem concerns functions that are not themselves elementary, but satisfy equations composed only of elementary functions. For example, the function defined by the equation $e^w - w = x$ belongs to this class of what are usually termed "implicitly elementary" functions. In this case, substantial generalizations are possible, though the mathematics involved becomes significantly more complicated. The interested reader is referred, for example, to [6], which proves a 53 year-old conjecture of Ritt to the effect that if the integral of an elementary function is implicitly elementary, then it is elementary. Other generalizations of this sort are also possible.

Finally, it is interesting to conjecture why, after a gap of 130 years following Liouville's work, during which, apart from the work of Ritt and Ostrowski [7], hardly any literature appeared concerning the problem, (Hardy [8] commented that "[Liouville's memoirs] seem to have fallen into an oblivion which they certainly do not deserve") there was a sudden re-awakening of interest in the 1960s and 1970s. It is no coincidence that during the last 20 years the exponential rise in computing power has made it possible to construct increasingly accomplished symbolic manipulation packages that can exploit the classical theorems to perform indefinite integration. An *algorithmic* approach to the problem of determining whether an elementary function has an elementary integral was discussed by Risch [9]. In principle, this provides a *complete* solution to the problem. However, the sophistication of the algorithm, combined with the fact that many of the integrals that computer algebra packages are commonly required to perform are of a fairly simple nature, render the full implementation of the Risch algorithm somewhat unwieldy and in practice the algorithm is usually "chopped" and combined with some sort of pattern search. The results of such packages are impressive. For example, the computer algebra package MAPLE took less than 1 second CPU time to determine that

$$\int \sqrt{\tan x}\ dx = \frac{1}{\sqrt{2}}\left(\arctan\left(\frac{\sqrt{2\tan x}}{1-\tan x}\right) - \log\left(\frac{1+\tan x+\sqrt{2\tan x}}{\sqrt{1+\tan^2 x}}\right)\right).$$

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Carpet bowls. A set of 8 bowls (measuring 66m diameter) and a jack, enabling you to play indoors – in an area as small as 4m × 1½m! From the BCA catalogue, sent in by John Round.

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"What about PI and E", said one irate reader; "If we accept PI and E, neither of which is finite, then could we not include I which signifies the square root of 1?" "N is used in mathematics to stand for any, and therefore every number". From *The Sunday Times* of 4, 11 and 18 April, sent in by Peter Ransom.