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AN INTEGRAL EQUATION FOR THE VALUE OF A STOP-LOSS OPTION

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ABSTRACT. The value of an investment may be protected by a 'stop-loss' option, where the underlying is sold if ever its price drops to a given fraction of its previously realized maximum price. Expressions for the fair value of a stop-loss option are derived for both the cases of continuous and discrete observations of the maximum price. In the continuous case, a simple exact solution is available, and when the observations are discrete the problem reduces to an integral equation. In some limiting cases this equation may be solved analytically, but in general a numerical solution is required and some indications are given of how this may be accomplished.

1. Definition of the Option

Options and other related derivative products may be used in a number of different ways, some of which may be for 'insurance' purposes (as in portfolio hedging) whilst others may be purely speculative. We describe a stop-loss option that mimics the following loss minimization strategy. Consider a situation where an underlying asset is held and we wish to insure our position should a sudden downturn in the market occur. Suppose that at time t the particular underlying has price $S(t)$ and we define $J(t) = \max S(\tau)$ for $\tau < t$, then one way of guarding against such a devaluation is to instruct a broker to sell the underlying if ever the price drops to such an extent that $S \leq \lambda J$, where $\lambda < 1$ is given. For obvious reasons, we refer to this as a 'stop-loss' strategy. A stop-loss option is an option that has the same financial payoff as this strategy, though lacking the dividend or coupon payments of the underlying. The option has no expiry date and is only exercised if S falls to the value λJ. Invoking the language of exotic options, we see that the stop-loss option may be thought of as a perpetual American barrier lookback option. Our aim is to determine the fair value for such a contract.

2. The Stop-loss option with continuous sampling

Firstly we examine the case where it is assumed that the maximum underlying price $J(t)$ is monitored continuously, and, if necessary, changed with every 'tick' of the stock market. Assuming that the volatility, interest rate and (constant) dividend yield are given by σ, r and D respectively, the equation governing the value $V(S,J,t)$ of the option may be derived using the standard Black-Scholes argument. Recall that for simple (i.e. non path-dependent) options the value is given by the solution of the Black-Scholes equation (see [2] and [5])

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + (r-D)SV_S - rV = 0. \tag{1}$$

In the present case this is still the governing equation, and J only occurs through the boundary conditions. Also, in contrast to other 'exotic' options, the equation includes no $\partial V/\partial J$ term. (See [3].) When the maximum is sampled continuously, and in view of the perpetual nature of the option, it is reasonable to assume that the option value has reached a steady state. In practical terms, this assumption is valid provided the option has been held for a long time. We must therefore solve (1) without the V_t term. To derive suitable boundary conditions for the equation, we note that $V(\lambda J, J) = \lambda J$ since if ever $S = \lambda J$ then the underlying will be sold and will have the value λJ. Additionally, V_J is zero on $S = J$. This reflects the fact that on $S = J$, V cannot depend on J, since if S ever reaches a previous J during its random walk, then, with probability 1, J will be exceeded at some later time. The equation to be solved is thus

$$\frac{\sigma^2 S^2}{2} V_{SS} + (r-D)SV_S - rV = 0 \qquad (\lambda J < S < J) \tag{2}$$

with

$$V(\lambda J, J) = \lambda J \qquad \text{and} \qquad V_J = 0 \quad \text{on} \quad S = J. \tag{3}$$

This problem is of similarity type; seeking a solution of the form $V = JW(\eta)$ where $\eta = S/J$ we find that

$$\frac{\sigma^2 \eta^2}{2} W'' + (r-D)\eta W' - rW = 0$$

and

$$W(1) = W'(1), \qquad W(\lambda) = \lambda.$$

Hence there is a simple explicit solution given by

$$W = \frac{\lambda[(k_2-1)\eta^{k_1} + (1-k_1)\eta^{k_2}]}{(k_2-1)\lambda^{k_1} + (1-k_1)\lambda^{k_2}} \tag{4}$$

where k_1 and k_2 are defined by

$$k_{1,2} = \frac{D + \sigma^2/2 - r \pm \sqrt{(r - D - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2},$$

taking the positive and negative signs respectively. In the particular case when there are no dividends, we find that $W = \eta$ and therefore $V = S$, so that the option value and underlying price are identical. For non-zero dividends however, (assuming that $D \leq r$) the value of the option is always less than the underlying, since the underlying yields a dividend but the option does not. Figure (1) shows W as a function of η for typical values of the parameters, namely $\sigma = 0.3$, $r = 0.08$, $\lambda = 0.5$ and $D = 0, 0.05, (0.01)$.

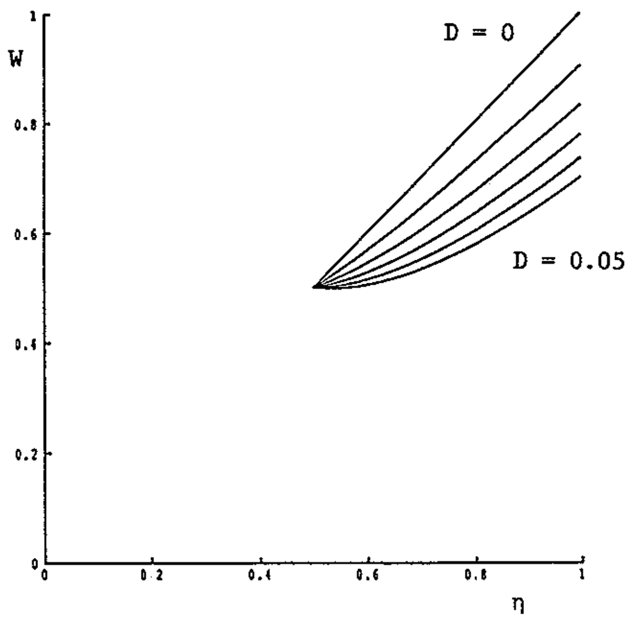

Figure (1) : Option values for various dividends

3. Discrete Sampling of the Maximum

In practice it is likely to prove inconvenient to monitor the price of the underlying in a continuous fashion, and it is more realistic to consider the case where only discrete measurements of the maximum price $J(t)$ are made. Assuming that the measurements are made at times t_i $(i = 0, 1, ...)$, a Black-Scholes type analysis shows that the equation that must be solved is (1), but we no longer have $S < J$. The price of the underlying, S, can exceed the most recently measured maximum since this maximum is only updated in a discrete fashion. The first of the boundary conditions given by (3) still applies, but for discrete sampling the second must be dropped. This second boundary condition is replaced by a limit on the growth of V as $S \to \infty$: we must have $V_{SS} \to 0$ as $S \to \infty$. Although this is a similar problem to the case of continuous sampling, the option value is now intrinsically time-dependent because of the discrete sampling. Since the value of J may jump at the sampling dates t_i we may expect the option price to jump there also. In fact, the option price for fixed S and J will jump across each sampling date but the *realized* value of the option will not be discontinuous. This is apparent from arbitrage considerations. Since the realized option value must be continuous when J is discontinuous we arrive at the jump condition

$$V(S, J, t_i^-) = V(S, \max(S, J), t_i^+),$$

where t_i^- and t_i^+ represent times immediately before and after the sampling respectively.

To simplify the problem it is now convenient to change from independent variables (S, t) to dimensionless variables (y, τ) where

$$S = \lambda J e^y, \qquad t = -2\tau/\sigma^2.$$

We then define a new dependent variable $\Omega(y, \tau)$ by

$$V(S, t) = J\lambda e^{\alpha t + \beta y}\Omega(y, \tau)$$

where α and β are to be chosen. With this change of variables, it is found that the choices

$$\alpha = r + \frac{\sigma^2\beta^2}{2}, \qquad \beta = \frac{1}{2} + \frac{D - r}{\sigma^2}$$

are particularly convenient, since Ω then satisfies the diffusion equation

$$\Omega_\tau = \Omega_{yy}.$$

The relevant boundary condition is now

$$\Omega = \exp(2\alpha\tau/\sigma^2) \quad \text{on} \quad y = 0$$

whilst the transformed jump condition is

$$\Omega(y, \tau_i^-) = [\max(\lambda e^y, 1)]^{1-\beta}\Omega(y - \log[\max(\lambda e^y, 1)], \tau_i^+). \tag{5}$$

This, along with the assumption of perpetuity and a suitable finiteness condition as $y \to \infty$, gives a well-posed problem. To solve this problem, (we assume without loss of generality that $\tau_i > 0$ for all i) it is convenient to set $\Omega(y, \tau_i^-) = \Phi(y)$, and derive an integral equation for $\Phi(y)$. First the diffusion equation must be solved for Ω. The solution having the correct growth for large y and satisfying $\Omega = \exp(2\alpha\tau/\sigma^2)$ on $y = 0$ and $\Omega(y, 0) = \Phi(y)$ is given by

$$\Omega = \frac{1}{2\sqrt{\pi\tau}} \int_0^\infty \left[\exp\left(-\frac{(y-s)^2}{4\tau}\right) - \exp\left(-\frac{(y+s)^2}{4\tau}\right)\right] \Phi(s) ds +$$

$$\frac{1}{2}\exp\left(\frac{2\alpha\tau}{\sigma^2}\right)\left[e^{-y\sqrt{\frac{2\alpha}{\sigma^2}}}\operatorname{erfc}\left(\frac{y}{2\sqrt{\tau}} - \sqrt{\frac{2\alpha\tau}{\sigma^2}}\right) + e^{y\sqrt{\frac{2\alpha}{\sigma^2}}}\operatorname{erfc}\left(\frac{y}{2\sqrt{\tau}} + \sqrt{\frac{2\alpha\tau}{\sigma^2}}\right)\right].$$

The jump and periodicity conditions must now be applied. To do this, we note that because of the form of (5), there will be no jump for values of y satisfying $0 \le y \le -\log\lambda$. The financial interpretation of this statement is that there is no jump in the option value unless a new maximum is sampled. We therefore have

$$\Omega(y, \tau_i^-) = \begin{cases} \lambda^{1-\beta} e^{y(1-\beta)} \Omega(-\log\lambda, \tau_i^+) & (y \ge -\log\lambda) \\ \Omega(y, \tau_i^+) & (0 \le y \le -\log\lambda) \end{cases}$$

Although any distribution of the observation times t_i can be handled using the theory given above, for illustrative purposes let us assume that observations of the maximum are performed periodically with period T; in practice T will typically be a day or a week—sampling at the close of trading daily or weekly. Consequently $\Phi(y)$ satisfies

$$\Phi(y) = \Phi(-\log\lambda) e^{(y+\log\lambda)(1-\beta)}$$

for $y \ge -\log\lambda$ and, using the periodicity condition $V(S, 0) = V(S, T)$

$$\Phi(y) = \frac{e^{-\frac{2\alpha T}{\sigma^2}}}{2\sqrt{\pi T}} \int_0^\infty \left(e^{-\frac{(y-s)^2}{4T}} - e^{-\frac{(y+s)^2}{4T}}\right) \Phi(s) ds +$$

$$\frac{1}{2}\left(e^{-y\sqrt{\frac{2\alpha}{\sigma^2}}}\operatorname{erfc}\left[\frac{y}{2\sqrt{T}} - \sqrt{\frac{2\alpha T}{\sigma^2}}\right] + e^{y\sqrt{\frac{2\alpha}{\sigma^2}}}\operatorname{erfc}\left[\frac{y}{2\sqrt{T}} + \sqrt{\frac{2\alpha T}{\sigma^2}}\right]\right)$$

for $0 \le y \le -\log\lambda$. This is a linear Fredholm integral equation of the second kind over a finite range, and may be further simplified by setting

$$\Phi(y) = e^{-ky} + \phi(y)$$

where $k^2 = 2\alpha/\sigma^2$, giving

$$\phi(y) = \frac{e^{-k^2 T}}{2\sqrt{\pi T}} \int_0^\infty \left(e^{-\frac{(y-s)^2}{4T}} - e^{-\frac{(y+s)^2}{4T}}\right) \phi(s) ds \qquad (0 \le y \le -\log\lambda) \qquad (6)$$

so that $\phi(0) = 0$ and therefore $\Phi(0) = 1$.

Before considering the solution of the integral equation (6) for arbitrary T, we may consider some special cases: for small values of T, standard applications of Laplace's method and Watson's lemma (see, for example [4]) show that, as $T \to 0$,

$$\int_0^\infty \exp\left(-\frac{(y-s)^2}{4T}\right)\phi(s)ds \sim \phi(y)2\sqrt{\pi T} + \phi''(y)2\sqrt{\pi T^3} + \phi^{(iv)}(y)\sqrt{\pi T^5} + O(T^3)$$

and

$$\int_0^\infty \exp\left(-\frac{(y+s)^2}{4T}\right)\phi(s)ds \sim e^{-y^2/4T}\left[\frac{2T}{y} + \frac{4T^2}{y^2}\phi'(0) + O(T^3)\right].$$

Contributions from the second term in the kernel are thus exponentially small and, writing $\phi(y) = \phi_0(y) + T\phi_1(y)$ we see that as $T \to 0$ the function $\phi_0(y)$ must satisfy

$$\phi_0'' - k^2\phi_0 = 0$$

whilst the correction term satisfies

$$\phi_1'' - k^2\phi_1 = \frac{1}{2}(\phi_0^{(iv)} + k^2\phi_0'' - k^2\phi_0).$$

From (6) we see that one boundary condition for this equation is given by $\phi(0) = 0$, so that, restricting our attention to the leading order term, we find that $\phi_0(y) = A\sinh ky$ where A is a constant. To determine A, we must exploit the fact that, as $T \to 0$, the condition $\partial V/\partial J = 0$ when $S = J$ is recovered. This corresponds to the condition

$$(1-\beta)\Phi(-\log\lambda) = \Phi'(-\log\lambda)$$

and, noting that $k_1 = \beta + k$ and $k_2 = \beta - k$ we find that

$$A = \frac{(k_2 - 1)\lambda^{k_1}}{(k_2-1)\lambda^{k_1} + (1-k_1)\lambda^{k_2}}.$$

Now ϕ_0 may be used to show that the leading order contribution to Ω is

$$\Omega(y,\tau) = e^{k^2\tau}[(1-A)e^{-ky} + Ae^{ky}],$$

and from this V may easily be recovered and shown to be identical to (4). Moreover, the correction term for small T may be easily determined.

Some analysis of (6) may also be carried out to examine the limit $T \to \infty$, though this is a less interesting case since if the sampling of the maximum is very infrequent the value of the underlying is safeguarded only to a minimal extent. The Fredholm equation that results from this analysis possesses a separable kernel and may be solved using standard methods, but the conclusion is the expected one, namely that as $T \to \infty$ the fair value of the option becomes exponentially small.

We now discuss briefly the general case where the maximum is sampled periodically. Although (6) must be solved numerically, there are many methods for accomplishing this in an an accurate and efficient manner, and details may be found in [1]. Before numerical methods can be employed however, it is essential to have some confidence that the problem is well-defined for ϕ for all values of y. We have

$$\phi(y) = e^{(1-\beta)y}\lambda^{1-\beta}[\lambda^k + \phi(-\log\lambda)] - e^{-ky}, \qquad (y \geq -\log\lambda)$$

and so (setting $Y = -\log\lambda$) for $0 \leq y \leq Y$ the function $\phi(y)$ satisfies

$$\phi(y) = \int_0^Y K(y,s)\phi(s)ds + f_1(y)\phi(Y) + f_2(y) \tag{7}$$

where

$$f_1(y) = \int_Y^\infty \lambda^{1-\beta}K(y,s)e^{s(1-\beta)}ds, \quad f_2(y) = \int_Y^\infty K(y,s)[\lambda^{k+1-\beta}e^{s(1-\beta)} - e^{-ks}]ds,$$

$$K(y,s) = \frac{e^{-k^2T}}{2\sqrt{\pi T}}\left(e^{-\frac{(y-s)^2}{4T}} - e^{-\frac{(y+s)^2}{4T}}\right).$$

We observe that (7) is a linear second kind non-homogeneous Fredholm equation. It therefore possesses a unique solution (and if it does, $\phi(Y)$ is also uniquely determined and may be found simply by setting $y = Y$) so long as the corresponding problem with $f_2(y)$ set equal to zero has only the trivial solution $\phi(y) = 0$. That this condition pertains in the present case is easily shown, (space does not permit an exposition of all the details, but the required theory is contained in, for example, [6]) and is partly a consequence of the fact that the operator associated with $K(y,s)$ is compact, self-adjoint and bounded. The problem is thus well-defined and the value of the option may be calculated for any value of T.

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