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Asymptotics of slow flow of very small exponent power-law shear-thinning fluids in a wedge

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The incompressible slow viscous flow of a power-law shear-thinning fluid in a wedge-shaped region is considered in the specific instance where the stress is a very small power of the strain rate. Asymptotic analysis is used to determine the structure of similarity solutions. The flow is shown to possess an outer region with boundary layers at the walls. The boundary layers have an intricate structure consisting of a transition layer $O(\epsilon)$ adjoining an inner layer $O(\epsilon \ln \epsilon)$, which further adjoins an inner-inner layer $O(\epsilon)$ next to the wall. Explicit solutions are found in all the regions and the existence of 'dead zones' in the flow is discussed.

1 Introduction

Viscous flow in a wedge with a source or sink at the vertex has been studied in many different forms and for many different types of fluid. For a linear viscous fluid ('Jeffrey–Hamel flow') it has long been known that there is a critical wedge angle beyond which pure inflow or outflow solutions are impossible, and regions of reversed flow are inevitably present. (See, for example, Rosenhead, 1963.) This is essentially a consequence of the inertial terms in the equations, and for non-inertial flow no such maximum wedge angle exists.

In this paper we consider inflow in a wedge (i.e. flow towards the wedge apex) for the case of a non-Newtonian fluid, specifically a shear-thinning power-law fluid. In the practical problem that motivated this study (see below) the flow is slow, and inertial terms may be neglected. Such flows have been analysed elsewhere for general exponents, but in many cases of interest the power law exponent is small. Despite their numerous practical applications, these flows have hitherto been analysed only by numerical methods. In this study, we determine completely the asymptotic structure of the limit of the small-exponent flow problem. It transpires that the solution possesses an intricate boundary layer structure comprising an outer region, a very thin transition layer, a thin inner layer, and a very thin inner-inner layer.

The original motivation for this problem was provided by an extrusion problem originally posed by Corning Inc., NY, USA. To make a part of a catalytic converter, a clay suspension is extruded through a plate punctured with small holes. Because of the properties of the clay suspension, the flow towards the punctured plate is non-uniform. In particular, flow rates through neighbouring holes of similar size can, in practice, become very different. This can lead to unwanted hole blockage, which has a detrimental effect on the quality of the final product. Observations of the flow of the clay towards the punctured plate revealed that there are regions where the suspension is essentially stationary. These regions are commonly referred to as 'dead zones', and are thought to be the underlying cause of hole blockage. The idealized geometry considered in the present study was chosen to gain an understanding of the fundamental behaviour of the flow of fluids with properties similar to the suspended clays in the extrusion problem described above.

2 Mathematical formulation of the problem

The literature is rich with various models for non-Newtonian fluids. Figure 1 shows schematic plots of stress τ against strain rate $\dot{\gamma}$ for various power-law fluids, so called because there is a power-law relation between stress and the strain rate. Such simple relationships (often based on experimental results) give a useful description of the behaviour of real fluids and render analysis tractable. The simplest example is provided by a linear relationship. In this case, the fluid is a conventional Newtonian fluid, and the slope of the line is identified with the viscosity of the material. Other power-law fluids may be classified in two types: those with exponent greater than unity (*shear-thickening*), and those with exponent between zero and unity (*shear-thinning*; sometimes referred to as *pseudoplastic*). A shear-thinning fluid exhibits the characteristic that the effective viscosity, defined as the local slope of the stress-strain curve, decreases as the material is sheared.

The power-law shear-thinning fluid is considered here, as this is closely representative of the suspended clays of interest to Corning Inc. Experimental observations have indicated that the material is well modelled by the shear-thinning law with an exponent of around $\frac{1}{10}$, and the smallness of this exponent motivated the present study. Many shear-thinning materials possess a yield stress beneath which there is no movement of the fluid; in this case, however, experimental results have indicated that the yield stress is negligible, and for this reason it is henceforth ignored. Shear-thinning power-law fluids with small exponents also arise in many other industrial processes of more general interest. Examples of this include polymer melts and highly sheared metals.

In one dimension the relevant constitutive law for a power-law fluid is normally written as

$$\tau = a|\dot{\gamma}|^{\epsilon-1}\dot{\gamma}$$

where a is a constant and for the Corning problem $\epsilon \sim \frac{1}{10}$.

In this study, we wish to consider more general fluid motions. Assuming that the fluid may be taken as incompressible, and the flow is sufficiently slow, with small representative Reynolds number, the equations of motion are

$$\operatorname{div} T = \mathbf{0}, \quad \operatorname{div} \boldsymbol{q} = 0$$

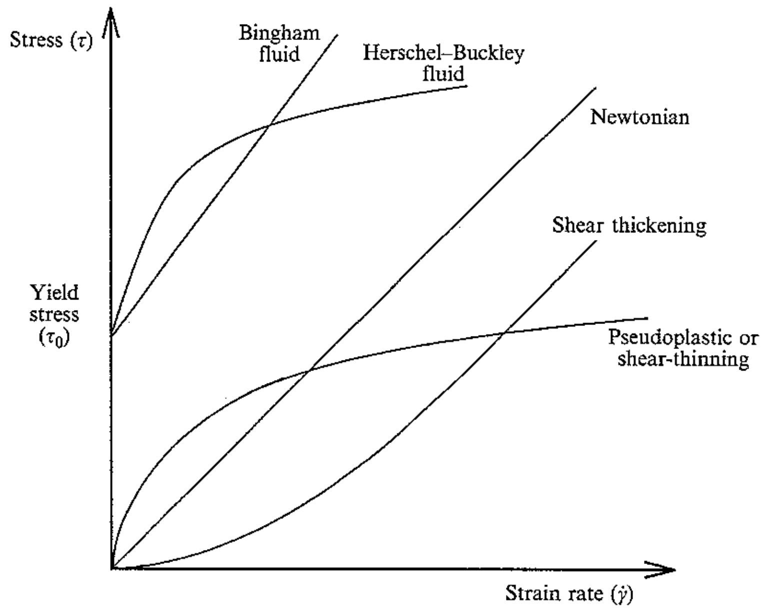

FIGURE 1 Stress plotted against strain rate for various different models of non-Newtonian fluids.

where T is the stress tensor and $\boldsymbol{q}$ is the velocity of the fluid. The stress tensor is given by

$$T_{ij} = -p\delta_{ij} + \tau_{ij}$$

where p is the pressure and the deviatoric stress τ_{ij} is given, by analogy to the earlier one-dimensional case, by

$$\tau_{ij} = a|(\dot{\gamma}_{kl}\dot{\gamma}_{kl})^{\frac{\epsilon-1}{2}}|\dot{\gamma}_{ij}$$

Here $\dot{\gamma}_{kl}$ is the infinitesimal strain tensor. For simplicity, the notation

$$K = |(\dot{\gamma}_{kl}\dot{\gamma}_{kl})^{\frac{\epsilon-1}{2}}|$$

will be used.

For wedge flow, we consider a cylindrical coordinate system (r, θ) centred at the wedge apex. Assuming that $\boldsymbol{q} = u\boldsymbol{e}_r + v\boldsymbol{e}_\theta$, where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are unit-vectors in the r and θ directions, respectively, the equations for slow viscous flow may be written

$$-p_r + \mu K\left(u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} - 2\frac{v_\theta}{r^2} - \frac{u}{r^2}\right) + 2\mu K_r u_r + \mu K_\theta\left(\frac{v_r}{r} + \frac{u_\theta}{r^2} - \frac{v}{r^2}\right) = 0,$$

$$-\frac{p_\theta}{r} + \mu K\left(v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} + 2\frac{u_\theta}{r^2} - \frac{v}{r^2}\right) + \mu K_r\left(v_r - \frac{v}{r} + \frac{u_\theta}{r}\right) - 2\mu K_\theta \frac{u_r}{r} = 0,$$

$$(ru)_r + v_\theta = 0,$$

where K is given by

$$K = \left[\frac{u^2}{r^2} + \frac{v^2}{2r^2} + \frac{u_\theta^2}{2r^2} + \frac{v_r^2}{2} + \frac{v_r u_\theta}{r} - \frac{v u_\theta}{r^2} + u_r^2 + \frac{2uv_\theta}{r^2} - \frac{v v_r}{r} + \frac{v_\theta^2}{r^2}\right]^{\frac{\epsilon-1}{2}}.$$

These equations are to be solved in the wedge region $r > 0$, $-\alpha < \theta < \alpha$ with boundary conditions

$$u = v = 0 \quad \text{on} \quad \theta = -\alpha, \quad \alpha$$

and a given total volume flux q defined by

$$\int_{-\alpha}^{\alpha} ru\, d\theta = q$$

We shall exploit the symmetry of the problem and only seek symmetric solutions.

The problem of slow viscous flow of power-law fluids has been considered elsewhere, and it is well known that similarity solutions exist for the wedge flow described above. A study of flow of a shear-thinning fluid between two intersecting planes was carried out by Mansutti & Rajagopal (1991), who sought solutions of the form $u = F(\theta)/r, v = 0$. They solved the resulting ordinary differential equations numerically for arbitrary wedge angles, but were unable to obtain solutions for small values of ϵ. A further study by Mansutti & Pontrelli (1991) extended the numerical analysis to cover the case $\epsilon \to 0$ by using an improved numerical technique. Their results indicate that boundary layer type structures exist at the wall. These boundary layers are artifacts of the nonlinearity of the fluid. We note that, for high Reynolds number flows, boundary layers of a more familiar form exist in regions near to the wall; a full study of such inertial boundary layers was carried out by Pakdemirli (1993).

In this paper we shall not consider these conventional inertia boundary layers, but concentrate on the boundary layer structure due to the nonlinear fluid properties suggested by the results of Mansutti & Pontrelli (1991) in the limit $\epsilon \to 0$.

2.1 Similarity solution

Following the previous work in this area, we seek simple similarity solutions of the form

$$u = \frac{F(\theta)}{r}, \quad v = 0.$$

Other, more involved similarity solutions, also exist, but we do not pursue these further here. Defining $m = (\epsilon - 1)/2$ and $\Phi = 2(F')^2 + 8F^2$, we have

$$K = \frac{\Phi^m}{2^{2m} r^{4m}},$$

and the equations of motion are

$$-p_r + \frac{\mu}{2^{2m} r^{4m+3}}[(\Phi^m F')' + 8m\Phi^m F] = 0, \tag{1}$$

$$-\frac{p_\theta}{r} + \frac{\mu}{2^{2m} r^{4m+3}}[2(\Phi^m F)' - 4m\Phi^m F'] = 0. \tag{2}$$

Elimination of the pressure by cross differentiation gives the equation

$$\Phi^m[F''' + (4 + 8m - 16m^2) F'] + (\Phi^m)'[2F'' + (4 + 16m) F] + (\Phi^m)'' F' = 0. \tag{3}$$

The boundary conditions appropriate to (3) are

$$F'(0) = 0, \tag{4}$$

$$F(\alpha) = 0, \tag{5}$$

$$\int_0^\alpha F(\theta)\, d\theta = q/2, \tag{6}$$

a prime denoting $d/d\theta$.

Since equation (3) is homogeneous, we may reduce its order by making the substitution $F = -e^G$. The minus sign has been included to ensure that $u < 0$, corresponding to the pure inflow problem; once the inflow problem has been solved then the form of the equations (1) and (2) ensures that the flow is fully reversible and thus covers the outflow problem. If G were allowed to take complex values, then it is possible that solutions containing both inflow and outflow could exist. We do not investigate such cases further, but speculate that, by analogy to the linear viscous fluid case, for purely radial slow flow there will be no cases where both inflow and outflow exist. If inertial terms were included, however, then it seems likely that, as in the Newtonian case, there may be a critical wedge angle beyond which both inflow and outflow must occur.

Setting $G' = f$, we find

$$(f^2+4)(4+\epsilon f^2)f'' - (1-\epsilon)(12+\epsilon f^2)f(f')^2 \\ +(4+f^2)ff'[4(1+2\epsilon^2)+\epsilon(1+2\epsilon)f^2]+\epsilon^2(4+f^2)^3 f = 0 \tag{7}$$

The boundary conditions (4), (5) become

$$G'(0) = 0, \tag{8}$$

$$G \to -\infty, \quad \text{as} \quad \theta \to \alpha, \tag{9}$$

and thus

$$f(0) = 0, \tag{10}$$

$$f = o\left(\frac{1}{\theta-\alpha}\right) \quad \text{as} \quad \theta \to \alpha. \tag{11}$$

The latter condition arises from the requirement that $G \to -\infty$. This condition is satisfied if (11) holds, and also if $f = O(1/(\theta-\alpha))$, so long as the constant of proportionality is strictly positive.

Once f is determined from (7), (10) and (11) the mass flow condition may be satisfied by choosing the arbitrary constant of integration in G. The problem for f may therefore be solved independently of q.

3 Asymptotic analysis

An approximate solution to (7), valid as $\epsilon \to 0$, will now be determined by the use of matched asymptotic expansions. Although the limit $\epsilon \to 0$ is not a singular perturbation in the classical sense (the order of equation (7) is not reduced by setting $\epsilon = 0$; see, for example, Nayfeh, 1973), the boundary condition (11) will force us to consider boundary layers, since terms that are negligible in the bulk or 'outer' flow, where f is order one, will become significant near the boundary, where f is large.

Below, it will be shown that there are four distinct regions of the flow, each corresponding to a different balance of terms in equation (7). In the 'outer' region away

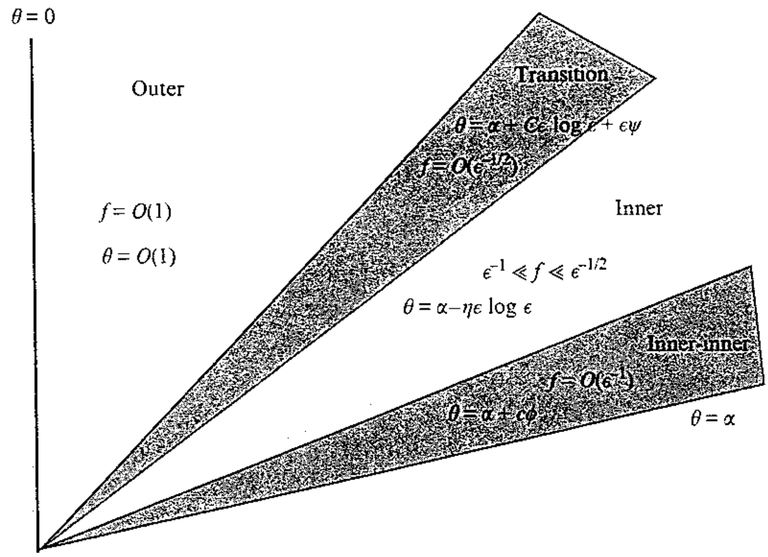

FIGURE 2. Schematic structure of boundary layers for wedge flow.

from the boundary, f and θ are $O(1)$. Adjoining this is a transition region of thickness $O(\epsilon)$, in which f is $O(\epsilon^{-1/2})$. The transition region matches onto an inner region of thickness $O(\epsilon \log \epsilon)$, in which f varies between $O(\epsilon^{-1/2})$ and $O(\epsilon^{-1})$. Finally, there is an inner-inner layer, of thickness $O(\epsilon)$, wherein f is $O(\epsilon^{-1})$. This structure is shown schematically in Fig. 2. We now discuss the solution in detail in each of the regions, matching adjoining regions together as we proceed. The analysis will be performed to generate the first-order solution in each region.

3.1 Outer region

Letting $\epsilon \to 0$ in (7) with θ and f order one, we find that the leading-order behaviour of f is given by

$$(4+f^2)f'' - 3f(f')^2 + (4+f^2)ff' = 0. \tag{12}$$

Dividing by $(4+f^2)^{5/2}$ turns this into an exact differential, which we integrate to obtain

$$A = \frac{2\sqrt{2}(4+f^2-f')}{(4+f^2)^{3/2}}, \tag{13}$$

where A is an arbitrary constant. The physical significance of A will be discussed later, but first we observe that, if the equation is written

$$f' = (4+f^2)\left[1 - \frac{A}{2\sqrt{2}}(4+f^2)^{1/2}\right],$$

then it is evident that, whenever $A > \sqrt{2}$, it is necessarily the case that $f' < 0$. For $A < \sqrt{2}$ we note that, since $f(0) = 0$, it must be the case that $f'(0) > 0$. Moreover, in this case it is clear that, if f was ever zero, then f' would be positive; thus for $A < \sqrt{2}$, solutions are strictly positive. In the case $A = \sqrt{2}$, the solution to the equation is simply $f = 0$. Since

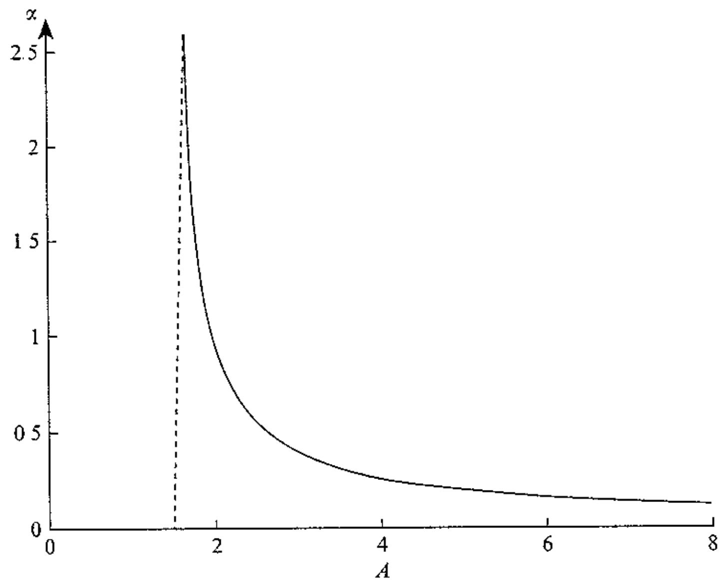

FIGURE 3. Relationship between wedge angle α and constant $A (A > \sqrt{2})$

we require f to tend to $-\infty$, it must therefore be the case that $A > \sqrt{2}$. Having established this, the equation may be integrated, giving the solution in the outer region as

$$\theta = \tan^{-1} w - \frac{A}{\sqrt{A^2-2}} \tan^{-1}\left(\sqrt{\frac{A+\sqrt{2}}{A-\sqrt{2}}}\, w\right) \tag{14}$$

where
$$w = \tanh\left(\frac{1}{2}\sinh^{-1}\frac{f}{2}\right),$$

the constant of integration having been determined by using the boundary condition $f(0) = 0$.

An examination of the behaviour of f as $\theta \to \alpha$ now shows that the condition (11) cannot be satisfied, suggesting that a boundary layer structure exists near $\theta = \alpha$ However, as $\theta \to \alpha$ we know that the solution must be large and negative, and the solution (14) can be made to satisfy this weaker condition. We therefore use the condition $f \to -\infty$ to determine A; once the boundary layer behaviour has been completely determined, it may be shown that this is, in fact, the correct matching condition into the boundary layer. Since $w = -1$ when $f = -\infty$, we find that

$$\alpha = \frac{A}{\sqrt{A^2-2}} \tan^{-1}\left(\sqrt{\frac{A+\sqrt{2}}{A-\sqrt{2}}}\right) - \frac{\pi}{4} \tag{15}$$

Figure 3 shows that relationship between A and α for $A > \sqrt{2}$. A unique value of A corresponds to each wedge semi-angle α for all α between 0 and ∞ To proceed further and determine the behaviour in the other regions, we also note that, as $\theta \to \alpha$,

$$f \sim -\frac{2^{1/4}}{\sqrt{A}\sqrt{\alpha-\theta}}. \tag{16}$$

We now examine the behaviour in the various regions comprising the boundary layer

3.2 Transition region

We introduce a new independent variable in the transition layer by setting

$$\theta = \alpha + C\epsilon \log \epsilon + \epsilon\psi. \tag{17}$$

The $O(1)$ constant C has been introduced to allow the transition region to be an $O(\epsilon \ln \epsilon)$ distance from the boundary $\theta = \alpha$, and the value of C will be determined in the course of the analysis The asymptotic behaviour (16) of the outer solution as it approaches the transition region motivates the rescaling of f according to

$$f = \frac{g}{\sqrt{\epsilon}} \tag{18}$$

Substituting (17) and (18) into (7), we find that at leading order

$$(4+g^2)gg'' - (12+g^2)(g')^2 = 0, \tag{19}$$

where prime denotes $d/d\psi$. Dividing by g^4, we again have an exact differential, which we may integrate twice to obtain

$$\log|g| - \frac{2}{g^2} = B\psi + D. \tag{20}$$

This expression is taken as defining the transition region solution

To match the outer and transition layer solutions, we write the transition solution in terms of the outer variables, and expand to obtain

$$f \sim \frac{g}{\sqrt{\epsilon}} \sim -\frac{\sqrt{2}}{\sqrt{\epsilon}\sqrt{-(B\psi+D)}} = -\frac{\sqrt{2}}{\sqrt{B}\sqrt{(\alpha-\theta+C\epsilon\log\epsilon-\epsilon D/B)}} \sim -\frac{\sqrt{2}}{\sqrt{B}\sqrt{\alpha-\theta}} \tag{21}$$

Matching this with the asymptotic behaviour of the outer solution as it approaches the transition layer

$$f \sim -\frac{2^{1/4}}{\sqrt{A}\sqrt{\alpha-\theta}},$$

we require that

$$B = \sqrt{2}A. \tag{22}$$

3.3 Inner region

Evidently, the transition layer solution does not satisfy the correct condition at the wedge boundary, though it does grow exponentially. We introduce an inner region where the order of f is not a specific power of ϵ, but rather varies between $O(\epsilon^{-1/2})$ and $O(\epsilon^{-1})$. Introducing a new independent variable in the inner region by setting

$$\theta = \alpha - \eta\epsilon \log \epsilon, \tag{23}$$

and assuming that

$$\frac{1}{\sqrt{\epsilon}} \ll f \ll \frac{1}{\epsilon}, \tag{24}$$

the leading-order behaviour of f is given by

$$ff''-(f')^2=0, \tag{25}$$

where prime now denotes $d/d\eta$. We write the solution as

$$f=\frac{b}{\epsilon}e^{a\eta\log\epsilon}, \tag{26}$$

where a and b are arbitrary constants (both assumed to be greater than zero) independent of ϵ. Then (24) holds providing $0 < a\eta < \frac{1}{2}$ and the expression (26) is the solution in the inner region.

To carry out the matching, we write the transition solution in terms of the inner variables and expand to obtain

$$f \sim \frac{g}{\sqrt{\epsilon}} \sim -\frac{e^{D}e^{B\psi}}{\sqrt{\epsilon}} = -\frac{e^{D}e^{-B(\eta+C)\log\epsilon}}{\sqrt{\epsilon}} = -\frac{e^{D}e^{-B\eta\log\epsilon}}{\epsilon^{1/2+\sqrt{2}AC}}. \tag{27}$$

Matching this with the inner solution we find

$$e^{D}=-b, \quad a=-B, \quad AC=\frac{1}{2\sqrt{2}}. \tag{28}$$

3.4 Inner-inner region

Finally, the inner region adjoins an inner-inner region where the wedge boundary condition may be satisfied Guided by the behaviour of the inner solution and the equation, we rescale f according to

$$f=\frac{h}{\epsilon} \tag{29}$$

and introduce a new independent variable in the inner-inner region by setting

$$\theta=\alpha+\epsilon\phi \quad (-\infty<\phi\leqslant 0). \tag{30}$$

Substituting (30) and (29) into (7), we find that at leading order

$$hh''-(h')^2+h^2h'=0, \tag{31}$$

where prime now denotes $d/d\phi$, with solution

$$h=\frac{c\,d\,e^{c\phi}}{1+d\,e^{c\phi}}. \tag{32}$$

The boundary condition (11) implies that $h(0)=-\infty$, and thus (assuming that $c>0$), $d=-1$. Consideration of the behaviour of h for small ϕ now shows that, in the vicinity of $\phi=0$, $h\sim 1/\phi$, and therefore $f\sim 1/(\theta-\alpha)$, so (11) is satisfied. The inner-inner solution is therefore

$$h=-\frac{c\,e^{c\phi}}{1-e^{c\phi}}$$

The matching is completed by writing the inner-inner solution in terms of the inner variables and expanding to obtain

$$f \sim \frac{h}{\epsilon} = -\frac{c\,\mathrm{e}^{-c\eta\log\epsilon}}{\epsilon(1-\mathrm{e}^{-c\eta\log\epsilon})} \sim -\frac{c\,\mathrm{e}^{-c\eta\log\epsilon}}{\epsilon}. \tag{33}$$

Matching this with (26), we find that $-c = b$ and $-c = a$.

The constants in all the regions are thus given by

$$B = A\sqrt{2}, \quad C = \frac{1}{2A\sqrt{2}}, \quad a = -A\sqrt{2}, \quad b = -A\sqrt{2}, \quad c = A\sqrt{2}, \quad D = \log(A\sqrt{2}),$$

so that all the required sign constraints are satisfied if A is positive.

4 Pressure field and flow rate

It is of interest to examine the relationship between the pressure gradient, the volume flux q, and the constant A Writing equations (1) and (2) in terms of f and G, we find

$$\frac{2^{\frac{\epsilon-1}{2}} r^{2\epsilon+1} p_r}{\mu} = -[f(f^2+4)^{\frac{\epsilon-1}{2}} \mathrm{e}^{\epsilon G}]' + 4(1-\epsilon)(f^2+4)^{\frac{\epsilon-1}{2}} \mathrm{e}^{\epsilon G}, \tag{34}$$

$$\frac{2^{\frac{\epsilon-1}{2}} r^{2\epsilon} p_\theta}{\mu} = -[2(f^2+4)^{\frac{\epsilon-1}{2}} \mathrm{e}^{\epsilon G}]' + 2(\epsilon-1) f(f^2+4)^{\frac{\epsilon-1}{2}} \mathrm{e}^{\epsilon G}. \tag{35}$$

In the limit as $\epsilon \to 0$, with G order one, we find

$$\frac{r p_r}{\sqrt{2}\mu} = -[f(f^2+4)^{-1/2}]' + 4(f^2+4)^{-1/2}, \tag{36}$$

$$\frac{p_\theta}{\sqrt{2}\mu} = -[2(f^2+4)^{-1/2}]' - 2f(f^2+4)^{-1/2} \tag{37}$$

Since p_θ is a function of θ only, it must be the case that rp_r is constant, and letting $P = rp_r/2\mu$ we see that

$$P = \frac{2\sqrt{2}(4+f^2-f')}{(4+f^2)^{3/2}} = A$$

Thus A is equal to the pressure gradient, when G (and hence the mass flux) is order one. Since for inflow $p_r > 0$ and thus $P > 0$, it is confirmed that $c > 0$ and $a < 0$, $b < 0$ as required earlier. Note also that in this case, the leading-order pressure gradient simply determines the order of the mass flux; the actual value of q is determined by the next order correction to the pressure gradient.

Since A has already been determined as a function of α, it follows that the pressure gradient is determined for a given flow rate and wedge angle. It is interesting to ask what the flow rate is for a given applied pressure gradient. When the applied pressure gradient is not equal to A, then the mass flux cannot be order one. We have mentioned already that the mass flux condition is satisfied by the addition of a suitable constant to G. When P is

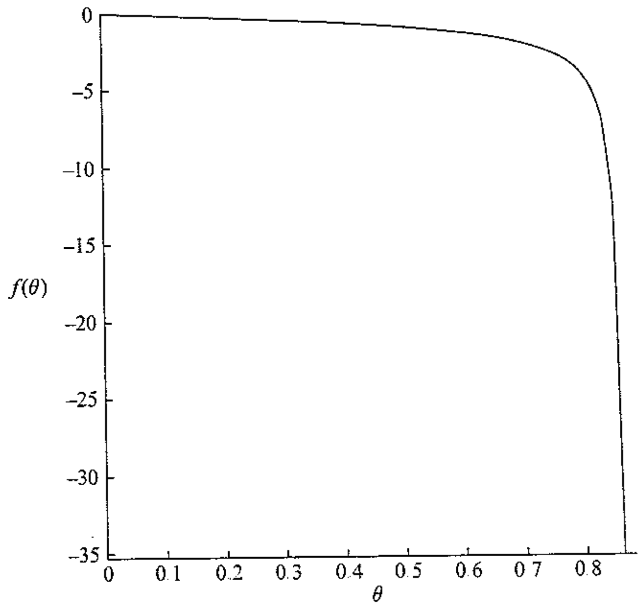

FIGURE 4. Plot of f as a function of angle θ for the case $\epsilon = \frac{1}{20}$, $A = 2$ (wedge semi angle of $\alpha \sim 0.881$).

not equal to A this constant is not order one, but order $1/\epsilon$. From (34) we see that if $G = G_0/\epsilon + G_1$, where G_0 is constant, then $P = A\,\mathrm{e}^{G_0}$, and hence

$$G_0 = \log(P/A)$$

Thus the velocity everywhere is simply premultiplied by a constant. For values of $P < A$ this constant, and hence the mass flux, is exponentially small in ϵ, while for values of $P > A$ it is exponentially large.

Figure 4 shows the variation of f with angle θ for the case $\epsilon = 1/20$ and $A = 2$, corresponding to a wedge semi-angle of $\alpha = 0.881 \sim 50.5$ degrees. The construction of a composite expansion for plotting purposes is greatly simplified by forming the composite expansion of the outer solution given by (14) and the solution $f(\theta)$ of the nonlinear equation

$$\log\left(\frac{A\sqrt{2\epsilon}f}{\epsilon f - A\sqrt{2}}\right) - \frac{2}{\epsilon f^2} = \frac{A\sqrt{2}(\theta-\alpha)}{\epsilon} + \log\left(\frac{A\sqrt{2}}{\sqrt{\epsilon}}\right) \qquad (38)$$

The equation (38) reproduces the correct asymptotic form of the solution in the transition, inner and inner-inner regions, and may be economically solved using a library routine.

It is worth mentioning that an alternative composite expansion may be constructed by noting from the matched asymptotic expansion analysis that certain terms in the governing equation (7) never enter the problem to leading order. As a result of this, it is possible to derive a solution that is uniformly valid in all regions. The easiest way to express this solution is as the integral

$$\int_{-\infty}^{f} \frac{(1+\epsilon)(4+\epsilon t^2)\,dt}{(4+t^2)\,[(\epsilon t^2-4)+A(4+t^2)^{\frac{1-\epsilon}{2}}]} = \alpha - \theta.$$

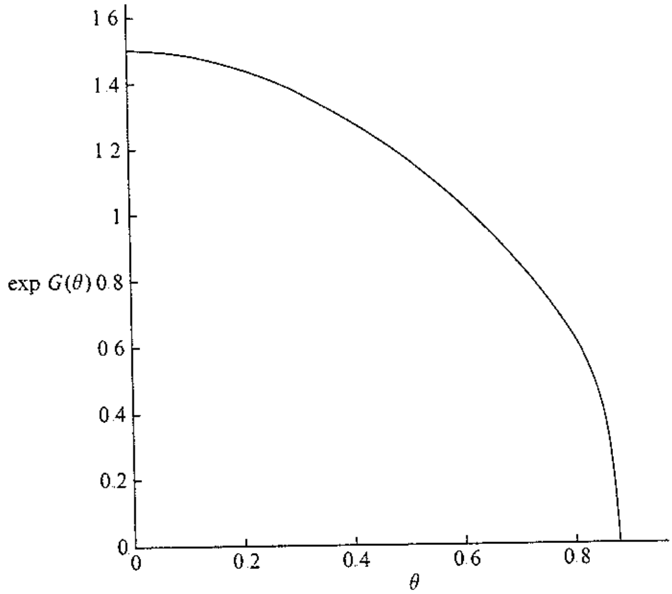

FIGURE 5. Plot of the velocity $-u/r = \exp(G(\theta))$ for the case $\epsilon = \frac{1}{20}$, $A = 2$ (data as for Fig. 4).

This expression may be used to confirm the asymptotic structure that has been derived above.

In Fig. 5 the velocity $\exp(G(\theta))$ is shown for the same data as Fig. 4. The integration required to recover g from f was carried out using a simple trapezoidal rule, and the additive constant in G was chosen to ensure that the total volume flux q was unity.

Finally, the asymptotic solution was calculated for the values $A = 1.9121$ (corresponding to a wedge semi-angle of 1 radian) and $\epsilon = 0.001$. With the volume flux chosen to be 2, it was found that $\exp(G(0))$ took the value 1.29, which agrees with the numerical solutions determined for this case by Mansutti & Pontrelli (1991). (Some care is required in this comparison as a factor of 2 has been omitted in Mansutti and Pontrelli's paper.) It may easily be verified that the previous numerical results are reproduced for arbitrary values of θ. Mansutti and Pontrelli also carried out numerical calculations for the same geometry using different values of ϵ, and comparisons between the asymptotic and numerical calculations for these cases also show good agreement. Even when ϵ is as large as $\frac{1}{2}$, the asymptotic solution is still satisfactorily accurate, predicting a value of $\exp(G(0))$ of 1.56, compared to the numerical result of 1.63.

5 Discussion

The analysis presented above has shown the basic structure of a power-law shear-thinning fluid flow adjacent to a fixed wall for the special case of a wedge. It is important to identify the resulting intricate structure, since in practically significant flows it is found that there are 'dead regions' where velocities are extremely small, and flow conditions may cause malfunctions. Only by understanding the basic structure of the velocity in simple cases such as wedge geometries may progress be made in identifying the conditions under which industrial processes may fail.

Although the results above have been presented in isolation, they may have some interesting implications for non-Newtonian flows in general. Although beyond the scope of the current paper, it would, in particular, be interesting to compare the dependence of flow rate upon pressure gradient to that seen in other fluids. Alternative models such as a Bingham fluid, a Herschel–Bulkley fluid, and a double viscosity fluid where the viscosity is piecewise constant, changing at a critical shear rate could all be considered, and in addition to flow rate comparisons the wall shear and the thickness of wall slip regions could be calculated. Comparison with experimental results is also required.

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