The numerical and analytical solution of ill-posed systems of conservation laws

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Recently, much attention has been given to the study of mixed systems of conservation laws, in which evolutionary systems of partial differential equations have the property that some of their eigenvalues are complex This has led to some confusion, particularly in the field of two-phase flow, in which the correct form of the governing equations for different flow regimes is not clear. In this study we consider two mixed systems, one being a 2 × 2 system in which the analytic solution is known if certain special waves are defined and the other a prototype system of equations for modelling single-pressure two-phase flow. By using these examples it is shown both analytically and by numerical experiment that solving such sets of equations is far from an easy matter The results have implications for the modelling of two-phase flows and other mixed systems, suggesting that although in some cases it might be possible to calculate solutions successfully, great care is generally needed in interpreting numerical results. This emphasizes the continuing requirement for more detailed mathematical modelling of two-phase flows.

Keywords: two-phase flow, mixed systems of conservation laws, numerical methods for hyperbolic systems

Introduction

In recent years there has been a great deal of interest in the mathematical modelling and properties of "mixed" systems of conservation laws in which there are regions in phase space in which the eigenvalues of the system become zero, coincident, or complex. Such systems occur in many different branches of applied mathematics, two (of many) examples being the modelling of traffic flow[1] and the study of the Riemann problem for a van der Waals fluid.[2] Mixed systems have also been studied as prototype systems in their own right. Examples of such work are the papers by Shearer,[3] Keyfitz and Kranzer,[4] and Holden.[5] Although much is now known about the properties of such systems, little attention has been given to the problem of solving them numerically. As far as the modelling involved in such systems is concerned, mathematical opinion seems to be divided roughly into two camps, one of which refuses to entertain any mixed system of conservation laws, arguing that any modelling that has led to such a system must be incorrect, and the other of which accepts such systems seemingly without qualms. The purpose of the present study is to show that while the former point of view is rather limiting and leaves us with many difficult modelling problems to solve before we can address a wide range of physical problems, the latter point of view can be somewhat dangerous.

As far as the field of two-phase flow is concerned, the existence of regions of phase space where the eigenvalues become complex has been known for some time. A considerable literature exists concerning the modelling of such problems, but on the whole it is fair to say that the modelling problem has not completely been resolved. Stuhmiller[6] considered an incompressible two-phase flow model in which the interfacial pressure was not assumed equal to the bulk pressure, equality of pressures being an assumption that had been made in earlier models and was known to lead to the existence of complex characteristics. By adding drag and virtual mass terms to the equations of motion Stuhmiller was able to produce a set of conservation laws whose characteristics were real in the limit when the volume fraction of the dispersed phase tended to zero. It is known, however, that even for quite modest dispersed phase volume fractions, there are still complex eigenvalues. Hancox et al.[7] also considered two-pressure models, distinguishing between different two-phase flow regimes by including different interfacial pressure relationships. Their model for bubbly flow is essentially similar to Stuhmiller's. Again real characteristics were found in some limits of small dispersed phase volume fraction.

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Various two-pressure models for stratified flow were developed by Ransom and Hicks,[8] who supplemented the standard momentum and mass conservation laws by adding a void fraction propagation equation. Real characteristics were obtained, but the physical motivation of the void fraction propagation equation seems unclear, and the equation cannot be derived via any rational averaging means. Other criticism of models of this sort have also been made on the grounds that for steady flow along a pipe with a pressure drop a steady-state solution does not exist. Prosperetti and Van Wijngaarden[9] proposed an interesting model for two-phase bubbly flow that gave real characteristics in certain circumstances as long as the relative velocity was small. They included a somewhat ad hoc "relative velocity equation," however, and again their model suffers from the defect that the equations cannot be derived from any rational averaging process.

A review of existing two-phase flow models was undertaken by Stewart and Wendroff.[10] As well as standard two-phase models, they also considered so-called "barycentric" models in which some equations for the motion of the mixture were added to the system and models in which the velocities of the two phases are assumed equal (or proportional to each other with a known constant of proportionality), so that a total momentum equation is required. They also commented on the fact that, in general, existing models either seem to lead to complex characteristics in some circumstances or are nonphysical in some respects. Their comments on the numerical solution of nonhyperbolic problems suggest that, as far as they are concerned, there is no completely acceptable totally hyperbolic two-phase flow model.

From the above discussion it seems clear that although some modelling progress has been made, the problem is far from solved. Although in some very specialized flow regimes the eigenvalues may be real, there are still many cases in which the appearance of complex eigenvalues seems inevitable. Clearly, there is interest in studying mixed two-phase flow systems and relevant prototype systems, both from an analytical and a computational point of view.

After making some comments of a more theoretical nature we wish to illustrate some of the problems inherent in solving mixed systems by purely numerical means by considering two distinct sets of conservation laws. One set is a "p-system" with an elliptic region, which we are able to solve analytically (provided that certain compromises are made) as well as numerically, and the other is a prototype mixed system of equations for two-phase flow, whose analytical solution is not available.

Mathematical modelling of two-phase flow and mixed systems

A mixed "p-system"

Although the real motivation for considering sets of mixed-type conservation laws is the study of two-phase flow models, it is instructive to consider first a simpler mixed problem that admits an analytical solution. In particular we will further simplify matters by considering only Riemann problems, so the initial data we choose will consist simply of the specification of a "left" and a "right" state, both of which are constant and are separated by a notional "membrane," which is punctured at time $t = 0$ allowing the "flow" to begin. Such problems may be thought of as the simplest case of the more general Cauchy problem in which piecewise continuous data are given at $t = 0$. The particular system that we shall consider first is of the type denoted by Smoller[11] a "p-system" and appears in conservation form as

$$\mathbf{w}_t + \mathbf{F}(\mathbf{w})_x = 0 \qquad (1)$$

where

$$\mathbf{w} = (u, v)^T \qquad \mathbf{F} = (v^2/2, -u)^T \qquad (2)$$

The solution of the Riemann problem for these equations may now proceed much as in the case of a standard totally hyperbolic 2×2 system (for details, see, for example, Ref. 11): It is easily established that the eigenvalues of the system are given by

$$\lambda_1 = -(-v)^{1/2} \le 0 \le (-v)^{1/2} = \lambda_2$$

and are therefore real and distinct for $v < 0$, equal and zero for $v = 0$, and pure complex for $v > 0$. The associated unit right eigenvectors are

$$\mathbf{r}_1 = (1 - v)^{-1/2}((-v)^{1/2}, 1)^T$$
$$\mathbf{r}_2 = (1 - v)^{-1/2}(-(-v)^{1/2}, 1)^T$$

and the Rankine-Hugoniot conditions require that for a shock with left state $(u_L, v_L)^T$ the states u and v satisfy

$$u = u_L \pm ((v_L - v)(v^2/2 - v_L^2/2))^{1/2}$$

We also note that in the case $v_L = 0$ we have $\nabla\lambda_j(\mathbf{w})\cdot\mathbf{r}_j(\mathbf{w}) = 0$ so that here the system is not genuinely nonlinear. This line is usually known as a fognal. As far as entropy conditions are concerned, the standard Lax condition may be extended in the usual way when elliptic regions are present (see, for example, Ref. 12) to give the Liu-Oleinik condition that

$$s(\mathbf{w}_L, \mathbf{w}_R) \le s(\mathbf{w}_L, \mathbf{w})$$

where s is the shock speed, for all $\mathbf{w}$ on the Hugoniot curve between $\mathbf{w}_L$ and $\mathbf{w}_R$. Having dealt with the shocks, we note that a j-rarefaction will be given by

$$\mathbf{w} = \begin{cases} \mathbf{w}_L & x/t < \lambda_j(\mathbf{w}_L) \\ \boldsymbol{\eta}(x, t) & \lambda_j(\mathbf{w}_L) \le x/t \le \lambda_j(\mathbf{w}_R) \\ \mathbf{w}_R & x/t > \lambda_j(\mathbf{w}_R) \end{cases}$$

where $\boldsymbol{\eta}$ satisfies

$$\boldsymbol{\eta}'(\xi) = \mathbf{r}_j(\boldsymbol{\eta}(\xi)) \qquad \boldsymbol{\eta}(\lambda_j(\mathbf{w}_L)) = \mathbf{w}_L$$
$$\boldsymbol{\eta}(\lambda_j(\mathbf{w}_R)) = \mathbf{w}_R$$

so that there are no rarefactions for $v > 0$ and the rarefaction curves for $v < 0$ are given by

$$u = u_L \pm \tfrac{2}{3}[(-v_L)^{3/2} - v^{3/2}]$$

(and on these curves, $v = -(x/t)^2$). Without giving all the details of the solution to the Riemann problem, which are somewhat long and complicated (for full details, see Ref. 13), we make the following observations.

1. The solution proceeds by connecting the left and right states to each other via a series of shocks and/or rarefactions, always ensuring that the Rankine-Hugoniot conditions are satisfied (so that the solution curve in phase space is a Hugoniot line) and the entropy conditions are met. In addition, there is the obvious requirement that shock or rarefaction waves contained in the solution must not overtake each other.
2. The solution divides into two main parts according to whether $\mathbf{w}_L$ is in the real ($v_L < 0$) or the complex ($v_L > 0$) region of phase space. Cases in which the left state lies on the fognal may be dealt with in a fairly obvious fashion once the solution is known in these two regions. Without loss of generality the u-ordinate of $\mathbf{w}_L$ may always be taken as zero, and we have also chosen to take the v-ordinate of the left state as ± 2. Phase space is then divided into the regions shown in *Figure 1* for different choices of $\mathbf{w}_L$.
3. The regions in *Figure 1* are bounded by various curves in phase space. For the specific $\mathbf{w}_L$, R_1, R_2, S_1, and S_2 are the 1- and 2-family rarefaction and shock Hugoniots, respectively, while F is the fognal line. The curve S_1^* represents the reflection of the S_1 curve in the fognal, which is the locus of the "last point" that can be reached via a 1-shock, while $\hat{R}_2$ is the last 2-rarefaction emanating from the fognal. Similarly the curves S_2' and R_2' are the last 2-shock and 2-rarefaction curves in the case in which $\mathbf{w}_L$ lies in the complex region of phase space. The straight line L' has equation $v = 2$, and L is its reflection in the fognal.

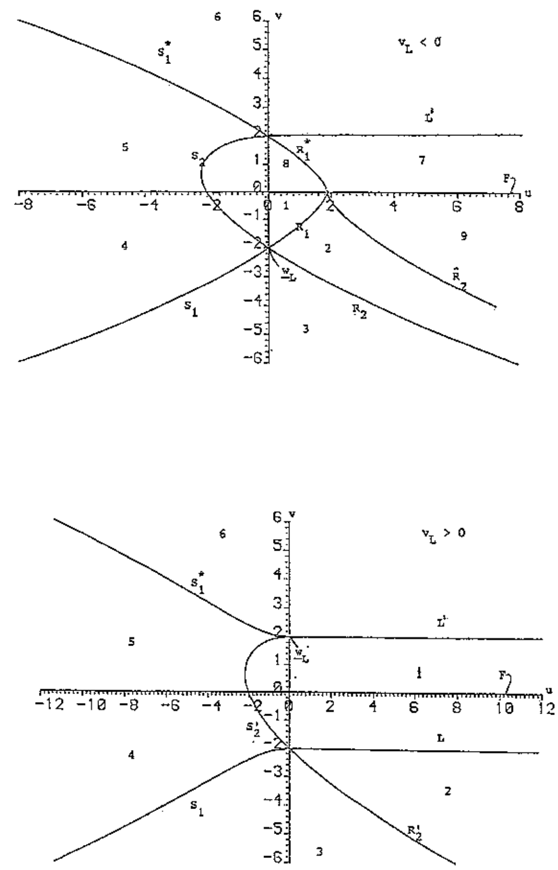

Figure 1. Hugoniot curves, lines, and regions for the mixed p-system for the cases $v_L < 0$, $v_L > 0$

If we denote successive intermediate states by $\mathbf{w}_I^n$ for $n = 1, 2 \ldots$, the solution of the Riemann problem may then be given as follows:

($v_L = -2$)

Region 1: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in R_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in S_2(\mathbf{w}_I^1)$
Region 2: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in R_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in R_2(\mathbf{w}_I^1)$
Region 3: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in R_2(\mathbf{w}_I^1)$
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Region 5: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in S_2(\mathbf{w}_I^1)$
Region 6: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_1^*(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in S_{sp}(\mathbf{w}_I^2)$
Region 7: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in R_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_2(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in S_{sp}(\mathbf{w}_I^2)$
Region 8: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in R_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in S_2(\mathbf{w}_I^1)$
Region 9: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in R_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_{sp}(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in R_2(\mathbf{w}_I^2)$

($v_L = 2$)

Region 1: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_{sp}(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_{st}(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in S_2(\mathbf{w}_I^2)$
Region 2: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_{sp}(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_{st}(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in R_2(\mathbf{w}_I^2)$
Region 3: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in R_2(\mathbf{w}_I^1)$
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Region 5: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_R \in S_2(\mathbf{w}_I^1)$
Region 6: $\mathbf{w}_L \longrightarrow \mathbf{w}_I^1 \in S_1(\mathbf{w}_L) \longrightarrow \mathbf{w}_I^2 \in S_1^*(\mathbf{w}_I^1) \longrightarrow \mathbf{w}_R \in S_{sp}(\mathbf{w}_I^2)$.

Here we have used a notation that is obvious for the most part but that involves some nonstandard solution components that require further discussion. Specifically, as well as shocks and rarefactions the (unique) solution employs *stationary* shocks and *monotonic split* shocks. These are described by the symbols S_{st} and S_{sp}, respectively. To give an example of the notation: For $v_L = -2$ and $\mathbf{w}_R$ in region 1 the left state is connected to an intermediate state $\mathbf{w}_I^1$ by a 1-rarefaction; $\mathbf{w}_I^1$ then joins to the right state $\mathbf{w}_R$ via a 2-shock to complete the solution.

The stationary and monotonic split shocks require some more explanation. A stationary shock is a shock of speed zero with $v_R = -v_L$, where $v_L > 0$. The zero speed may then be verified from the Rankine-Hugoniot relationships after elimination of u_L and u_R. The definition of a monotonic split shock has been made to allow a unique solution to be determined in all of the regions. Specifically, we allow only one of the Rankine-Hugoniot conditions to be satisfied in the normal way. The other is disregarded, since it gives rise to a shock of infinite speed. In general, therefore, for a monotonic split shock, u will change while v remains constant. Monotonic split shocks may therefore be considered to be "normal" to stationary shocks. The monotonicity is included in the definition for the obvious reasons of uniqueness. With these assumptions it is easy to prove the following results.

Proposition 1. For all $\mathbf{w}_L$ with $v_L < 0$, any connections to other states and further connections from these states to others must satisfy $v \le \max\{0, v_R\}$, where v_R is the v-ordinate of the final state.

This property may be used in the numerical calculations to help the stability of the computations. The following can also be shown.

Proposition 2. The Riemann problem may be solved uniquely for any right state when $v_L < 0$ by a solution consisting of rarefactions, standard shocks, and monotonic split shocks.

Proposition 3. When $v_L > 0$, the sets of points on the 1-shock curves are disconnected from $\mathbf{w}_L$ in phase space. Moreover, the solution contains no 1-rarefactions. The Riemann problem may be solved uniquely by a combination of shocks, 2-rarefactions, monotonic split shocks, and stationary shocks.

In each case it should be noted that the uniqueness is elementary but tedious to prove, amounting to nothing more than use of the definitions and the univalent covering properties of the shock and rarefaction curves. The cases in which $\mathbf{w}_L$ lies on the fognal may be treated similarly, except that now some of the regions become trivial.

The nonstandard features of the solution may be criticized on the grounds that they are somewhat artifical and in some senses contrary to what we expect from the solutions of hyperbolic problems. Nevertheless, we shall see that the numerics actually do a remarkable job in picking up these solution components. In the final analysis the only way of deciding whether or not some of the nonstandard features are "physical" is to consider a regularized system such as

$$\mathbf{y}_t + \mathbf{f}_x = \epsilon \mathbf{y}_{xx}$$

seeking travelling wave solutions of the form

$$\mathbf{y} = \mathbf{y}(\xi) \qquad (\xi = (x - st)/\epsilon)$$

where

$$\mathbf{y}(-\infty) = \mathbf{y}_L \qquad \mathbf{y}(+\infty) = \mathbf{y}_R \qquad \mathbf{y}'(\pm\infty) = 0$$

so determining the "viscous profile" that gives the shock as $\epsilon \to 0$ (for fuller details and more associate references, see Ref. 11). Such profiles, however, have proved notoriously hard to find for mixed problems, and the mathematical difficulties in locating them[3] seem to preclude any "physical" justification of this sort.

The two-phase flow equations

One of the criticisms that may be levelled against the p-system that we have considered is that it is only a 2×2 system and so either all of the eigenvalues are real or all are complex. Therefore we now consider a specific two-phase flow, namely gas/particulate flow, in which solid particles are dispersed in a gas. The solid particles are assumed to be large enough that the "dusty gas" approximation that is relevant in some cases and leads to real eigenvalues[14] is not valid. Moreover, the solid particles are themselves reactive and are continuously being converted into gas via burning. One application in which this variety of two-phase flow is particularly relevant is the internal ballistics of a large-caliber gun, in which the solid particles are gun propellant and the high-pressure gas that is produced by the combustion forces a projectile out of a barrel. Traditionally, in this field the single-pressure model has been used[15,16] that consists of the equations

$$(\rho_1 A_1)_t + (\rho_1 A_1 u_1)_x = C^{(1)}$$
$$(\rho_2 A_2)_t + (\rho_2 A_2 u_2)_x = C^{(2)}$$
$$(\rho_1 A_1 u_1)_t + (\rho_1 A_1 u_1^2)_x + A_1 p_x = C^{(3)}$$
$$(\rho_2 A_2 u_2)_t + (\rho_2 A_2 u_2^2)_x + A_2 p_x = C^{(4)}$$
$$(\rho_1 A_1 E_1)_t + \left[\rho_1 A_1 u_1 \left(E_1 + \frac{p}{\rho_1}\right)\right]_x + p(A_2 u_2)_x = C^{(5)}$$
$$(N_2)_t + (N_2 u_2)_x = 0$$

Here the subscript 1 refers to the gas phase and 2 to a solid phase, which we assume is made up of granular material; ρ represents density, u velocity, p pressure, and A cross-sectional area. For simplicity we take the total cross-sectional area to be unity so that $A_1 + A_2 = 1$. The total energy is given by $E = (u^2/2) + e$, where e is the internal energy, and N_2 denotes the number density of solid particles per unit length. Depending on the type of flow that we are considering, the source terms $C^{(1)}$ to $C^{(5)}$ may include details of interphase mass, momentum, and heat transfer as well as losses via conduction, diffusion, and other source terms, but we always assume that these source terms do not contain any derivatives. An equation of state for the gas phase must also be added to the model, and in many highly reactive flows this may have to be a second-order law such as the covolume equation. Here, however, for simplicity we assume that the gas is a perfect one and obeys the relationship

$$p = \rho_1 \mathfrak{R} T_1$$

where $\mathfrak{R}$ is the universal gas constant, and the sound speed c of the gas is given by $c^2 = \gamma p/\rho_1$. We also require a set of boundary conditions; for example, we always specify all of the flow variables at time $t = 0$ and assume that no flow takes place through any solid boundary.

Although, as remarked above, this model has traditionally been used for the study of internal ballistics in which the solid phase is assumed incompressible so that there is an energy equation only for the gas phase, it has also formed the basis of many other studies of widely differing two-phase flow regimes. Examples of this are the work of Thyagaraja et al.,[17] Assimacopoulos,[18] and Ardron and Duffey.[19] For the purposes of Riemann problem computations it is convenient to simplify the equations further. If we set the source terms to zero (since we assumed that the source terms contain no derivatives, this cannot affect the hyperbolicity of the system) and discount the equation for the number of particles, assuming this variable to be constant, then some simple algebra shows that the system may be recast as

$$\begin{aligned}
(\rho_1 A_1)_t + (u_1 \rho_1 A_1)_x &= 0\\
(A_2)_t + (u_2 A_2)_x &= 0\\
(u_1)_t + u_1 u_{1x} + \frac{p_x}{\rho_1} &= 0\\
(u_2)_t + \left(\frac{u_2^2}{2} + \frac{p}{\rho_2}\right)_x &= 0\\
(p)_t + u_1 p_x + \left(\frac{p\gamma}{A_1}\right)(u_1 A_1 + u_2 A_2)_x &= 0
\end{aligned} \tag{3}$$

and while this system is physically less realistic than the original one, the two are very similar mathematically. It should be noted that the equations are not in conservation form. This is a consequence of the averaging that has been used and is unavoidable. A standard characteristic analysis is easy to perform; writing the system as

$$\mathbf{w}_t + A\mathbf{w}_x = 0$$

where

$$\mathbf{w} = (\rho_1 A_1, A_2, u_1, u_2, p)^T$$

we find that the eigenvalues of A are given by

$$\lambda = yc + u_1$$

where the gas sound speed c is given by $c^2 = \gamma p/\rho_1$ and y satisfies the equation

$$y(y^4 - 2V^3 + y^2(V^2 - q - 1) + 2Vy - V^2) = 0 \tag{4}$$

with

$$V = \frac{u_2 - u_1}{c} \qquad q = \frac{A_2 \rho_1}{A_1 \rho_2}$$

Clearly, the zero characteristic speed corresponds to the incompressibility of the solid phase, but the behavior of the roots of the remaining quartic is less clear. For $V = 0$ (zero phasic relative velocity) we find trivially that there are two more zero and two real roots, but some elementary analysis shows that for nonzero V the equation has four real roots if and only if

$$V^2 > (1 + q^{1/3})^3$$

If this condition is not met, then there are two real and two complex sound speeds. The boundaries $V = 0$ and $q = 0$, where all of the characteristics are real, are therefore singular lines, and any perturbation of position in phase space will induce a pitchfork bifurcation from a repeated zero eigenvalue to two complex ones. Left eigenvectors for the system are given by

$$(-c^2, 0, 0, 0, 1) \quad \text{and} \quad \left(0, \frac{c^2 \rho_1 V_y}{A_1(V - y)}, c\rho_1, c\rho_2(y^2 - 1), y\right)$$

so that the Pfaffian ordinary differential equations for the Riemann invariants are

$$\begin{aligned}
&-c^2 d\rho_1 = dp\\
&\frac{c^2 \rho_1 V y}{A_1(V - y)} dA_1 + c\rho_1 du_1 + c\rho_2(y^2 - 1) du_2 + y\,dp = 0
\end{aligned}$$

The first of these is simply the definition of the sound speed, while in general the second is not integrable, so Riemann invariants cannot be explicitly derived and the problem solved completely as we were able to do for the p-system that was considered above. Indeed, the only case in which any of the nontrivial Riemann invariants may be determined is when $q = 0$. In this case the two eigenvalues $y = V$ give Riemann invariants $A_1 =$ constant, but this does not provide us with any more information. In spite of the impossibility of solving the general Riemann problem, the equations do have the helpful property that the eigenvalues depend only on the two parameters q and V. This means that the main features of the solution can be inferred from two-dimensional phase portraits, just as was possible for the problem (2).

As far as the physical motivation for the two systems of equations is concerned, it is probably best to treat (2) as merely a synthetic problem, though it can be interpreted as a nonlinear wave equation. Concerning the second system, it was noted in the introduction that the extent of the elliptic region in phase space may be decreased (though not, in general, eliminated) under certain circumstances by the addition to the model of extra derivative terms representing such effects as turbulence, drag, virtual mass, and torque. In view of the fact that the modelling problem has not been satisfactorily solved, we shall study the unextended system (3) because of its popularity and simplicity. It seems likely that similar conclusions will apply in elliptic regions of phase space to other versions of the equations (see, for example, Refs. 20–22).

Having described both of the conservation law systems that we wish to consider, some further comments on the role of "viscosity" are necessary as a prelude to numerical considerations. Although the particles in the flow are incompressible and the gas is assumed inviscid, is it the case that if a small amount of viscosity is added to the equations of motion, the complex characteristics will disappear? To address this question, consider instead of (2) the system of conservation laws

$$
\begin{aligned}
u_t + vv_x &= \epsilon q_x \\
v_t - u_x &= 0 \\
q &= u_x
\end{aligned}
\qquad (5)
$$

The eigenvalues of this 3×3 system are given by $\lambda = \infty, \infty$, and 0 so that the system is truly parabolic and no ill-posedness is present. In this respect the inviscid system is a singular limit of the viscous system. It is tempting to argue that this result shows that the inclusion of any nonzero viscosity makes the model well-posed, and therefore since some numerical diffusion will inevitably be introduced when the equations are solved numerically, the problems caused by the appearance of the complex characteristics in the inviscid model is a purely theoretical one. Unfortunately, this begs some important questions. First, we expect that when the flow is nearly inviscid, we will be able to use inviscid equations in the core of the flow, coupled to viscous boundary layer equations near the walls. This is a matter of experience, based on knowledge of single-phase flows in which the inviscid limit of the full equations is perfectly well-posed, and it would be very surprising if the same were not true for two-phase flow. Second, it is possible to demonstrate that the complex characteristics may taint the inviscid limit of the viscous equations.

Proposition 4. For the fully viscous version of the system (2) (that is, the system (2) with viscous terms added to both equations) there are always wave numbers k associated with Fourier modes that grow exponentially in time.

To show this, consider the effect of introducing a mode

$$\mathbf{w} = \mathbf{w}_0 e^{ikx - \omega t}$$

into the "frozen" system

$$A_0 \mathbf{w}_t + B_0 \mathbf{w}_x + C_0 \mathbf{w}_{xx} = 0$$

where $A_0 = I$, $C_0 = \text{diag}\,[-\epsilon, -\delta]$, and

$$B_0 = \begin{pmatrix} 0 & v_0 \\ -1 & 0 \end{pmatrix}$$

Here the subscript 0 represents the frozen state, and ϵ and δ are the two viscosities. For a nontrivial $\mathbf{w}_0$ we require that

$$\det[-\omega A + ikB - k^2 C] = 0$$

which gives

$$\omega = \frac{k^2(\epsilon + \delta) \pm [k^4(\epsilon + \delta)^2 - 4(\epsilon\delta k^4 - k^2 v_0)]^{1/2}}{2}$$

so that Re $(\omega) < 0$ whenever either $k^2(\epsilon + \delta) < 0$ or $v_0 > \epsilon\delta k^2$. The former situation can come about only for negative ϵ and δ, and this is no surprise—we know that the backward heat equation is ill-posed. The latter condition, however, can always arise for sufficiently large v_0. It is certainly true that if we draw the obvious parallel with the Euler equations and include viscosity only in the "momentum" equation ($\delta = 0$), then for any $v_0 > 0$ there will be exponentially growing modes. It may be argued that this has come about because of the somewhat pathological nature of the system (2), but it can also be shown (though the calculation involved is far more lengthy) that for the two-phase flow equations (3) with terms ϵu_{1xx} and δu_{2xx} added to the right-hand sides of the phase 1 and phase 2 momentum equations, respectively, the Fourier modes must satisfy $\omega = ik(u_1 + yc)$, where y is a solution of

$$
\begin{aligned}
&y^4 - 2Vy^3 + y^2(V^2 - q - 1) + 2yV - V^2 \\
&+ Ki\delta(y^3 - y + V - Vy^2) + \epsilon Kiy[V^2 + y^2 - q \\
&\qquad - 2Vy - K(V - y)\delta i] = 0 \qquad (6)
\end{aligned}
$$

Here $K = k/c$ and V and q are as defined in (4). We must now determine whether there are values of ω such that Re $(\omega) < 0$, and this is obviously a formidable task because of the complexity of (6). What we can say, however, is that for small viscosities where $\epsilon = O(\delta)$ and $K\delta << 1$ the solutions y of (6) will be given by

$$y = y_{inv} + O(K\delta)$$

where y_{inv} is the solution to (4). We know that (4) has solutions, however, with nonzero complex part, say of the form $a \pm ib$. This shows that

$$\omega = ik[u_1 + (a \pm ib)c]$$

and so there will inevitably be wave number k for which Re $(\omega) < 0$, and instability will result.

Far from being a deliberately pathological example, this system is regularly used for practical calculations. We conclude therefore that even for the viscous system the near-inviscid limit may suffer from ill-posedness.

Numerical methods for mixed problems

Having discussed two sets of conservation laws, we now wish to consider the status of existing numerical methods, for mixed problems. We begin by making an elementary observation that, though it has been made before, bears repetition.

Proposition 5. For a mixed system of conservation laws with complex eigenvalues in some part of phase space, a Fourier mode introduced into the solution will grow exponentially

The demonstration of this is easy: If we "freeze" the system of conservation laws

$$\mathbf{w}_t + A(\mathbf{w}, x, t)\mathbf{w}_x = 0$$

where A is the Jacobian of $\mathbf{f}$ at some time t_0 and position x_0, then by writing $A = PDP^{-1}$ in the normal way, where D is the diagonal matrix of eigenvalues and P the matrix of eigenvectors, the system may be uncoupled if we define $\mathbf{v}$ by $\mathbf{w} = P^{-1}\mathbf{v}$ to give in component form

$$(v_i)_t + \lambda_i(v_i)_x = 0$$

The x-dependence may be removed by taking a complex Fourier transform in x to give for the transform variable $V_i(k, t) = \mathcal{F}(v_i(x, t))$

$$(V_i)_t + \lambda_i ik V_i = 0$$

Clearly now if any of the eigenvalues have nonzero complex parts, then inevitably an exponentially growing Fourier mode will be introduced.

To investigate the effects that complex characteristics may have on any numerical calculations that we may make, consider the application of the Lax-Friedrichs method to the nonhomogeneous linear advection equation

$$u_t + au_x = f(u)$$

where a is a complex number. Approximating the source term by $f(u) = su$ (s real), we find after a standard stability analysis that the symbol of Lax-Friedrichs finite difference operator is given by

$$\rho(\xi) = (\tfrac{1}{2} + a\lambda/2)e^{-i\xi} + (\tfrac{1}{2} - a\lambda/2)e^{i\xi} + \frac{s}{2}(e^{-i\xi} + e^{i\xi})$$

In the case in which $s = 0$, a real, we simply retrieve the standard CFL condition for stability, which requires that $\lambda|a| < 1$, while if a is real and s is nonzero, the stability is not affected unless the source term is too strong. The conclusions when $\mathrm{Im}\,(a) \neq 0$ are somewhat different, however. For $s = 0$, $a = b + ic$, we find that

$$|\rho(\xi)|^2 \geq 1 + c\lambda \sin 2\xi - \sin^2 \xi$$

which always exceeds unity for some small value of ξ, whatever the value of c. This means that in strict terms the Lax-Friedrichs scheme is unconditionally unstable for any a with nonzero complex part. As an aside, we note that an analysis of the case in which a is complex and s is nonzero shows that under certain circumstances the source term may help the situation and make the scheme stable.

Also, it is no surprise to learn that similar conclusions may be drawn from the stability analysis of all of the well-known first-order monotone schemes. These stability results mean that numerical computation of solutions to mixed problems will be at least unreliable and at worst impossible. Further, it is easy to show that the total variation diminishing (TVD) property that schemes such as the Lax-Friedrichs and many other more complicated methods enjoy is now no longer operative. Thus oscillations and overshoots may be expected.

Some discussion of the relevance to TVD methods to two-phase flow calculations is necessary here. The development of TVD and other related methods in the past few years has lead to great advances in the numerical solution of hyperbolic problems, and it is now possible to capture shocks and contact discontinuities with great accuracy at a reasonable computational cost. (See, for example, Refs. 23–25.) It is also true that TVD methods remove many parasitic oscillations that are produced by standard second-order methods such as the Lax-Wendroff scheme. It should be remembered however that TVD methods (a) are only guaranteed oscillation-free for linear equations (small overshoots and oscillations are regularly present in the solutions of nonlinear problems), (b) do *not* necessarily produce the correct entropy solution,[26] and (c) are not applicable to mixed problems, since they require consideration of the eigenvalues and the eigenvectors, which may be complex. So while TVD methods work very well for solving hyperbolic problems, they are of little use for mixed problems. This means that for the numerical computations shown below, we are forced to employ a very simple method.

As a final comment, we note that although the outlook for the performance of standard finite difference methods is somewhat pessimistic, the situation is not completely hopeless. The stability analysis has shown only that there *might* be regions where the scheme becomes unstable. We have to hope that such modes do not enter our calculations. There is also the possibility that it might be possible to develop numerical methods for mixed problems that are capable of identifying these modes and damping them out. This is a topic of continuing research, but since this study is concerned with the status of standard calculation methods for mixed problems, we do not consider such methods further.

Numerical results

Having commented on both the modelling and numerical analysis of two mixed systems of conservation laws, we now wish to show that, while it is possible to solve mixed problems numerically in some circumstances, extreme care is required in interpreting the results. Some comments are apposite concerning the numerical method that was used for the calculations. For all of the results described below, the equation

$$\mathbf{w}_t + \mathbf{F}(\mathbf{w})_x = 0$$

was discretized simply as

$$\mathbf{w}_k^{n+1} = \tfrac{1}{2}(\mathbf{w}_{k+1}^n + \mathbf{w}_{k-1}^n) - \frac{dt}{2dx}(\mathbf{F}_{k+1}^n - \mathbf{F}_{k-1}^n)$$

Some of the reasons for using the extremely simple Lax-Friedrichs method were mentioned in the previous section. Additionally, it is worth noting that the method is monotone, first-order, and TVD, so for totally hyperbolic systems of equations we are guaranteed to pick up the weak solution that also satisfies both Rankine-Hugoniot and entropy conditions. Also, although we realize that in normal use the low-order accuracy of the method would produce unacceptable smearing of the shocks, in this case we are prepared to use a large number of mesh points (1000 for all of the results given below) and a high Courant number $\mathscr{C} = \max(\lambda_i)dt/dx$ (chosen to be 0.9 for all the computations) so that the shocks are captured cleanly. Another convenient property of the method is that it is symmetric, so we do not have to worry about the direction in which the shocks and rarefactions are travelling. Finally, the method is extremely simple This is important because we would not like our numerical conclusions to be clouded by the use of a more complicated method.

Considering first the numerical solution of various Riemann problems for equations (2), results are shown for the "membrane" dividing the two initial states of the Riemann problem situated at $x = 0.5$. In all of the figures, both the profiles of the spatial variation of the variables and the phase space portrait as the flow evolves may be seen. For the phase portraits we have followed the idea of Bell et al.,[27] who carried out Riemann problem studies on a system of conservation laws pertaining to the three-phase flow of oil, water, and gas in a porous medium by indicating individual points of phase space that have been passed through with a symbol as well as a line In this way the shocks are obvious, since the symbols are individually recognizable. For the rarefactions, however, the line appears virtually continuous. *Figure 2* shows the computed solution to the Riemann problem at time $t = 0.2$, where $\mathbf{w}_L = (0, -2)^T$ and $\mathbf{w}_R = (0, -5)^T \in$ region 3. The numerical solution is excellent, with the 1-shock/2-rarefaction structure clearly visible. It may easily be verified numerically that the Rankine-Hugoniot and entropy conditions are all satisfied. Solutions may also be calculated very successfully when the right state is in regions 1, 2, and 4. Now consider the case in which the right state is in region 5. *Figure 3* shows two examples of this at time $t = 0.1$ for right states of $(-4, 1)^T$ and $(-4, 1.25)^T$ (left state $\mathbf{w}_L = (0, -2)^T$). For the first case the numerical solution is close to the analytical one, but a nonmonotonicity is clearly visible in both variables. For $v_R = 1.25$ the situation has become much worse. Severe

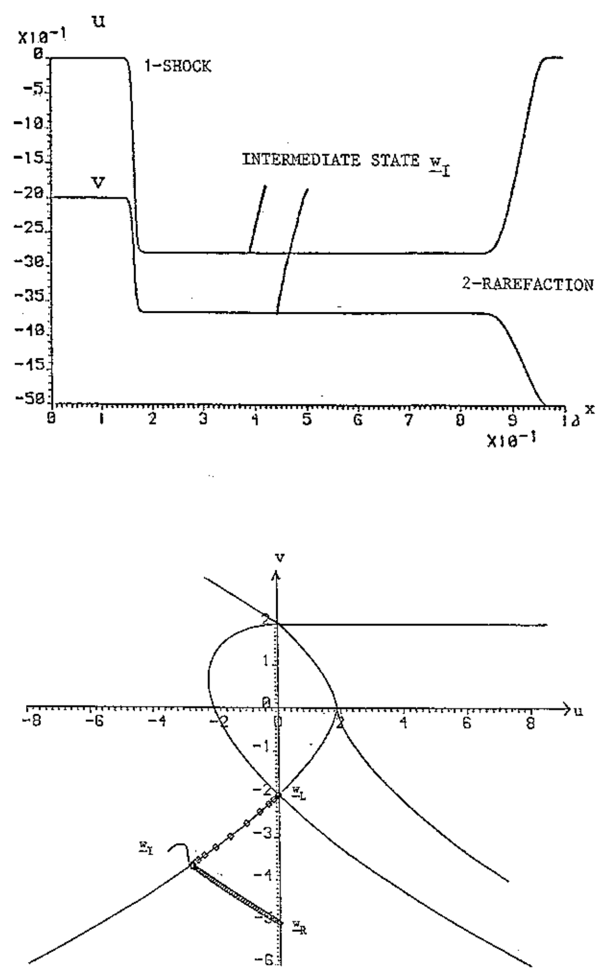

Figure 2. Profile plots and phase diagram for $\mathbf{w}_L = (0, -2)^T$, $\mathbf{w}_R$ in region 3 at time $t = 0.2$

oscillations are now present, and although the solution is still recognizable as being the one predicted by the theory, there are now many outlying points in the phase plane diagram that spoil the solution. In *Figure 4* we consider the right states $\mathbf{w}_R = (2, 6)^T$ and $\mathbf{w}_R = (5, -1)^T$, results being given at time $t = 0.1$. When necessary, we have also used the results of proposition 1 to keep the numerical solution under control. The expected solutions are well resolved and have clearly structured phase portraits.

Figure 5 shows two examples of the numerical solution when the left state is taken to be $\mathbf{w}_L = (0, 2)$. First, we consider the case $\mathbf{w}_R = (2, \frac{1}{2})^T$, where the solution contains a split shock, a stationary shock, and a 2-shock. The other case corresponds to $\mathbf{w}_R = (0, -5)^T$, so the initial Hugoniot curve is disconnected from the left state. In both cases the numerical solution is at least recognizable, though naturally the scheme experiences some difficulty in predicting the stationary shocks. Some oscillations are also present, but agreement on the whole is tolerable.

We now turn to the solution of the system (3). Once

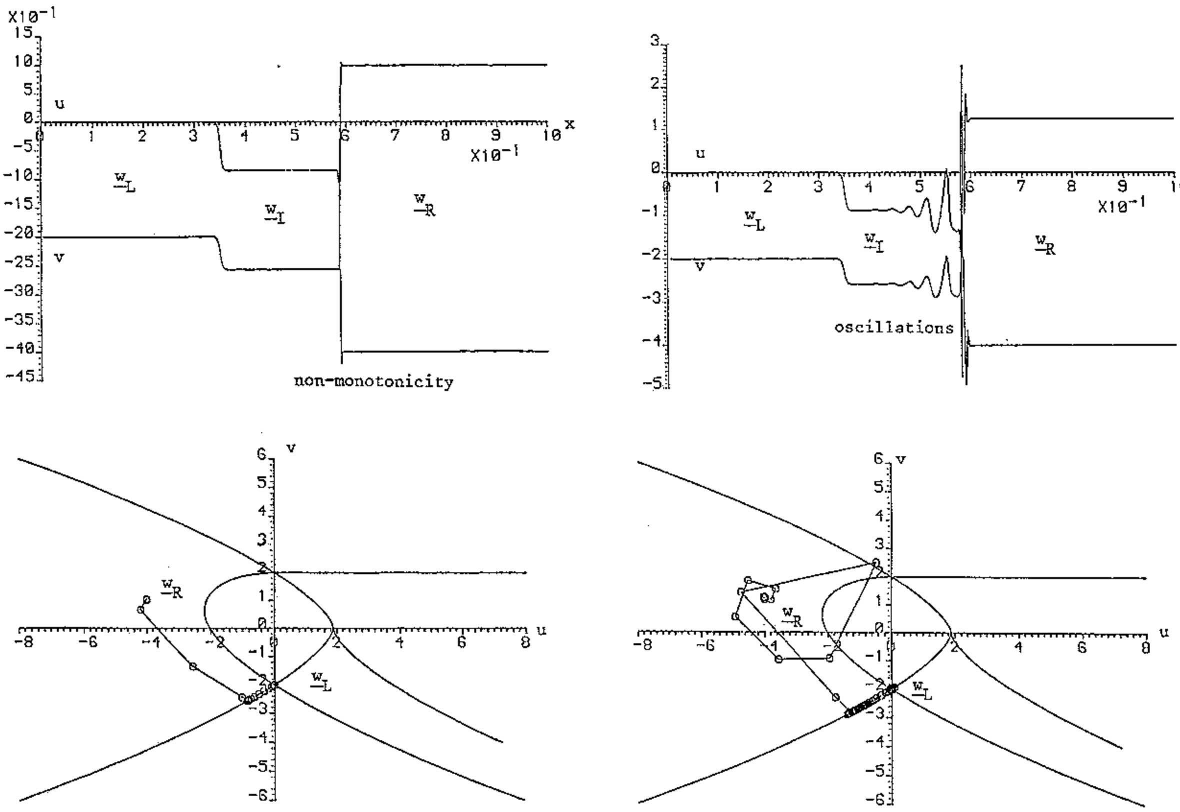

Figure 3. Profile plots and phase diagram for $\mathbf{w}_L = (0, -2)^T$, $\mathbf{w}_R = (-4, 1)^T$ and $(-4, -1\ 25)^T$ at time $t = 0.1$

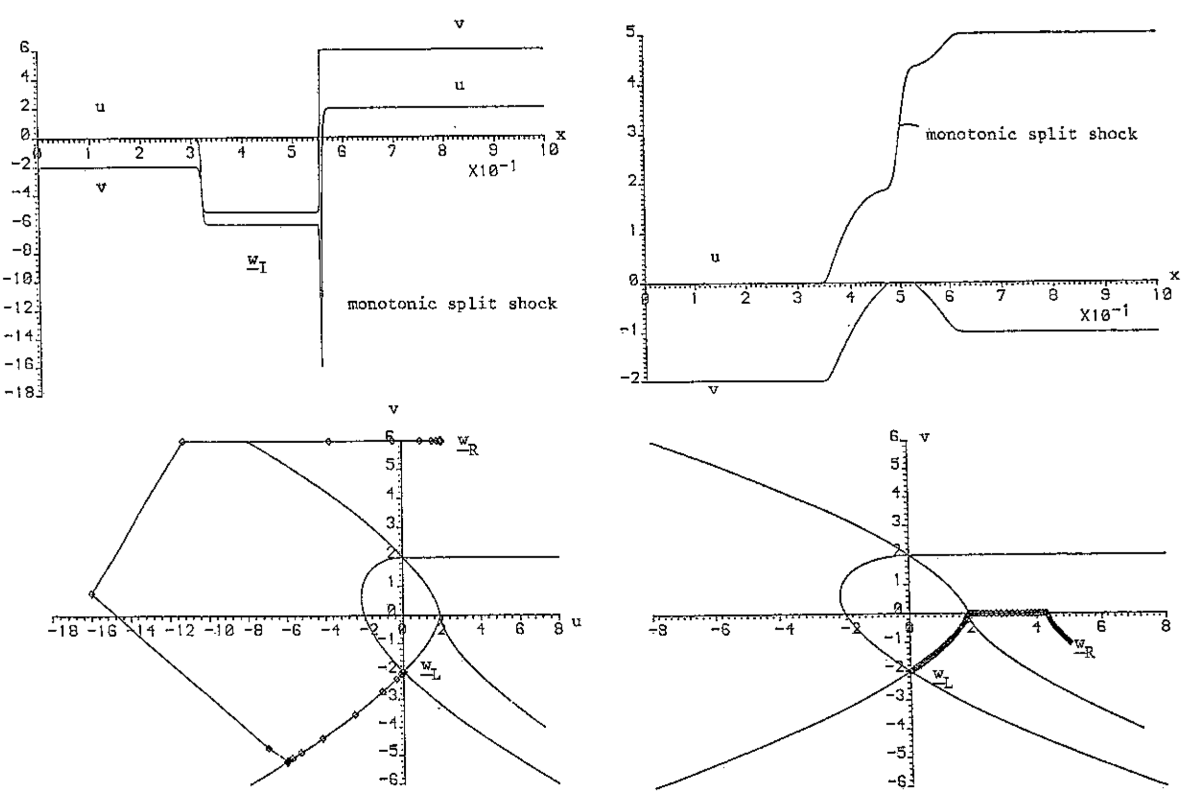

Figure 4. Profile plots and phase diagram for $\mathbf{w}_L = (0, -2)^T$ and $\mathbf{w}_R$ in regions 6 and 9, respectively, at time $t = 0.1$

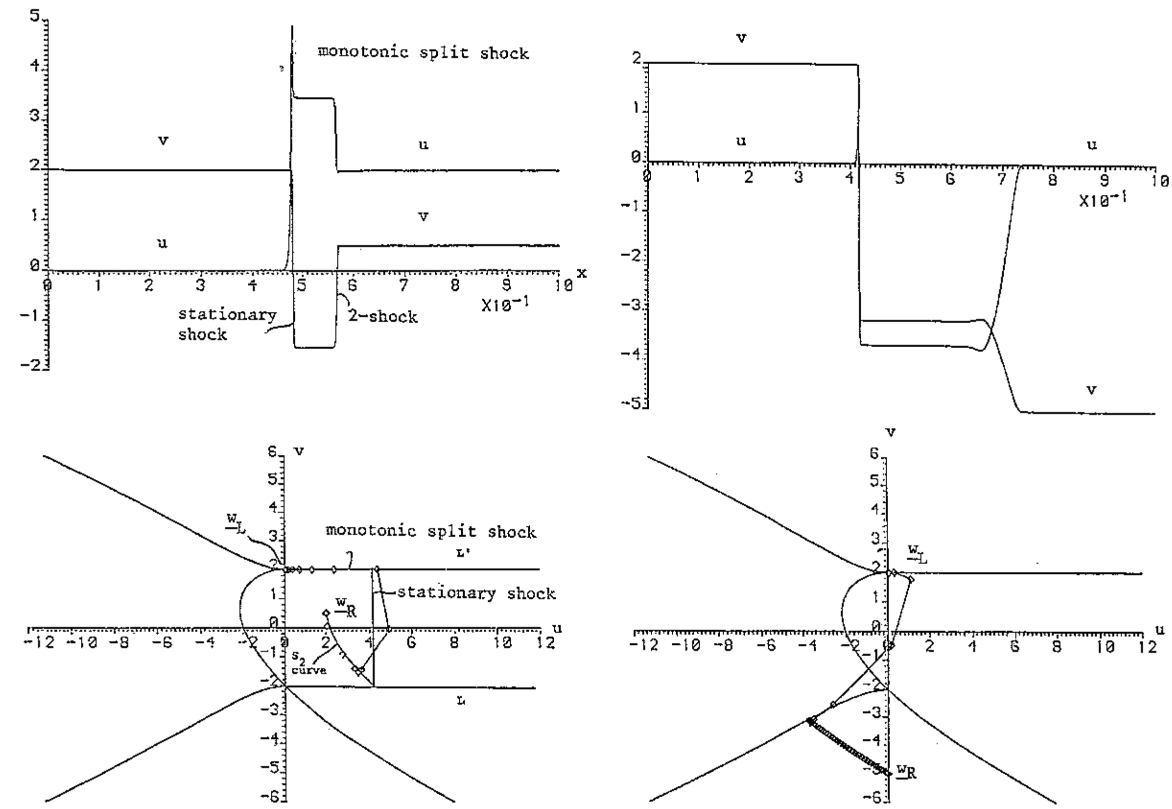

Figure 5. Profile plots and phase diagram for $\mathbf{w}_L = (0, -2)^T$ and $\mathbf{w}_R$ in regions 1 and 3, respectively, at time $t = 0.1$

again, 1000 points were used for the calculations, the Courant number was fixed at 0.9, and the "membrane" was situated at $x = 0.5$. *Figure 6* shows computations made for a Riemann problem with left and right states

$$\mathbf{w}_L = \begin{bmatrix} 0.1 \\ 0.5 \\ 5.0 \\ 2.0 \\ 0.214286 \end{bmatrix} \qquad \mathbf{w}_R = \begin{bmatrix} 0.05 \\ 0.5 \\ 5.0 \\ 2.0 \\ 0.128571 \end{bmatrix}$$

so that both states lie in the region where all of the characteristics are real. Both the profiles and the phase portrait in (q, V^2)-space are shown at time $t = 0.05$. The boundary between the region containing two complex eigenvalues and that containing all real eigenvalues is shown as a solid curve. It is apparent that the wave structure is very plausible. As in standard gas dynamics, the density exhibits two shocks and a "contact" discontinuity (somewhat smeared by the numerical scheme), and the pressure shows a wave system that consists of two shocks but no contact discontinuity. The phase plane portrait also seems quite normal, with no trespass into the complex characteristic region. There is also some evidence of a rarefaction wave structure close to the right state, but this is not clear enough to comment on with any degree of assuredness.

In *Figure 7* the left state and right states were chosen as

$$\mathbf{w}_L = \begin{bmatrix} 0.05 \\ 0.5 \\ 5.0 \\ 2.0 \\ 0.642857 \end{bmatrix} \qquad \mathbf{w}_R = \begin{bmatrix} 0.05 \\ 0.5 \\ 5.0 \\ 2.0 \\ 0.128571 \end{bmatrix}$$

so that while the right state was in the region of real characteristics (at the point $q = 0.1$, $V^2 = 5$), the left state was in the elliptic region with $q = 0.1$, $V^2 = 1$. Results are again shown at $t = 0.05$, and now the wave structure is much more complicated. The left state seems initially to undergo a rarefaction in which the pressure drops but the gas velocity rises. This is followed by a triple-shock system. A nonmonotonicity is observable in the pressure and the gas velocity between the second and third shocks. Although some elements of the numerical solution are not quite clear, in general the computations seem plausible.

Figure 8 shows a case in which the left and the right states are given by

$$\mathbf{w}_L = \begin{bmatrix} 0.64 \\ 0.2 \\ 5.0 \\ 2.0 \\ 5.14286 \end{bmatrix} \qquad \mathbf{w}_R = \begin{bmatrix} 0.05 \\ 0.3 \\ 6.0 \\ 2.04 \\ 0.128571 \end{bmatrix}$$

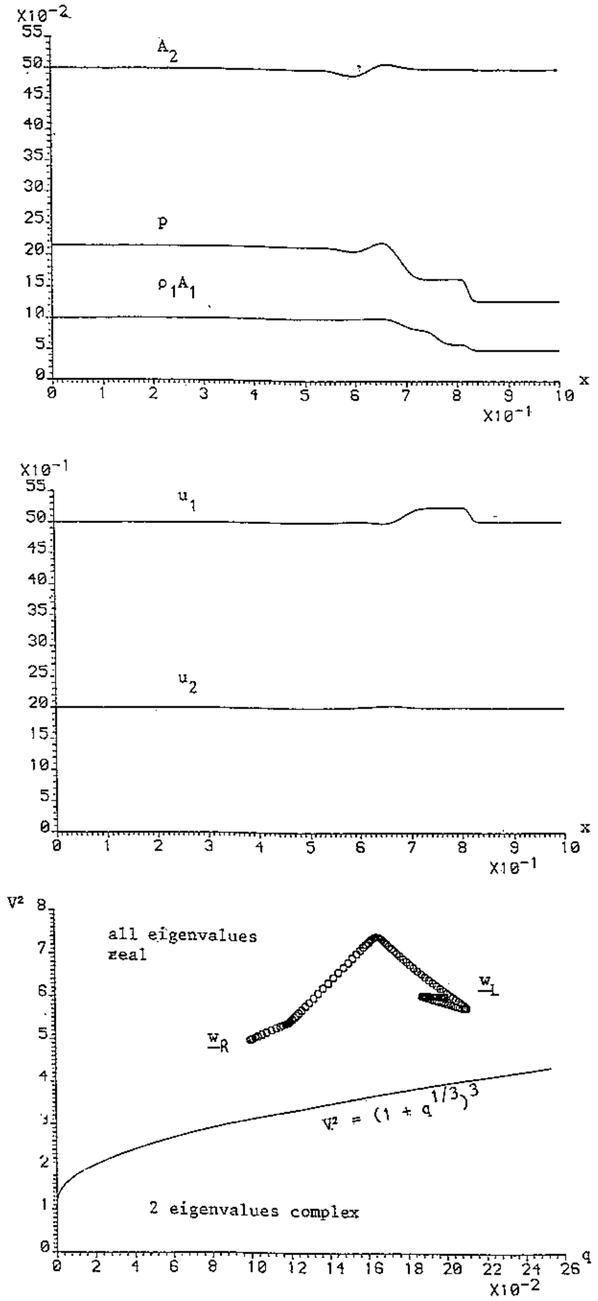

Figure 6. Two-phase flow: Profile plots and phase diagram for an example in which both states are in the totally hyperbolic region

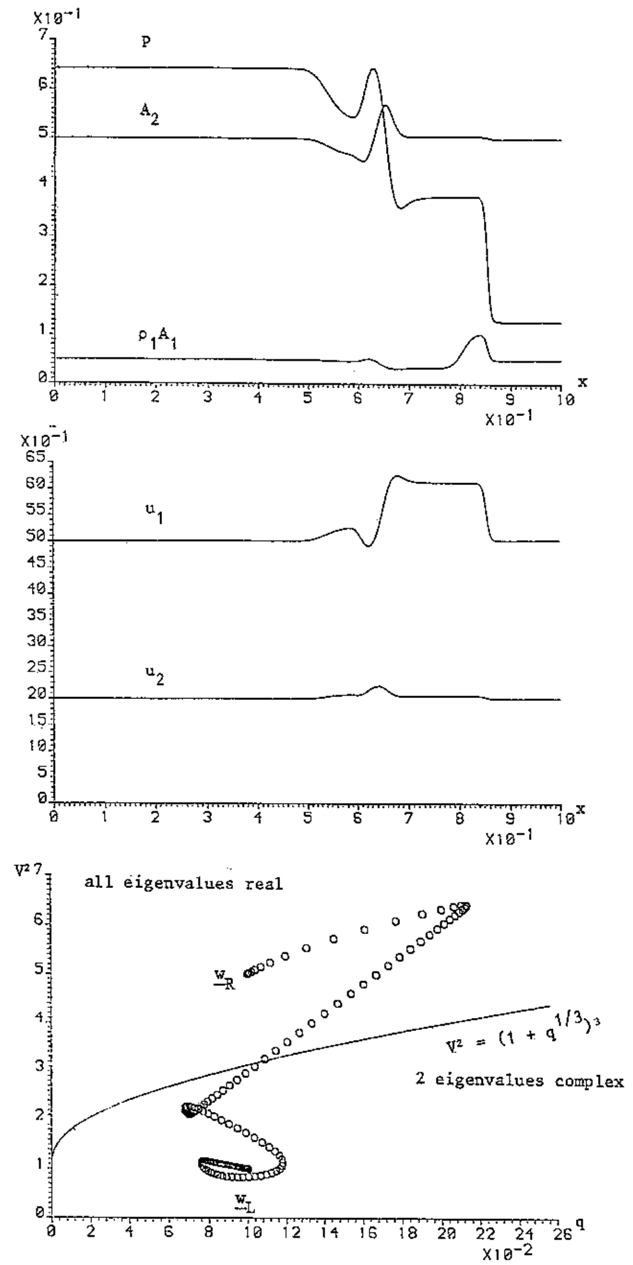

Figure 7. Two-phase flow: Profile plots and phase diagram for an example in which the left state is in the elliptic region and the right state is in the totally hyperbolic region

so the left state is in the elliptic region and the right state is in the real region. This problem may be thought of as a fairly realistic test problem in which a "hot wave" impinges on a two-phase system with initially uniform velocities. It should be noted that these physically reasonable data lead inevitably to a boundary state in the elliptic region. Indeed, if we were to consider any problem in which both phases were initially at rest, the model would have to admit phase paths that either entered into the elliptic region or took the unlikely course of penetrating into the totally hyperbolic region via the axes $V^2 = 0$ and then $q = 0$. Here the numerical results give cause for concern. The phase space portrait looks extremely suspicious, and the pressure profile in particular seems to be unphysical. Our worst fears are confirmed by *Figure 9*, in which, using the same left states as in the calculations for *Figure 8*, we consider a right state

$$\mathbf{w}_R = \mathbf{w}_L + \epsilon(1, 1, 1, 1, 1)^T \qquad (\epsilon << 1)$$

These data constitute a test of well-posedness, for it would seem very reasonable that for two states close

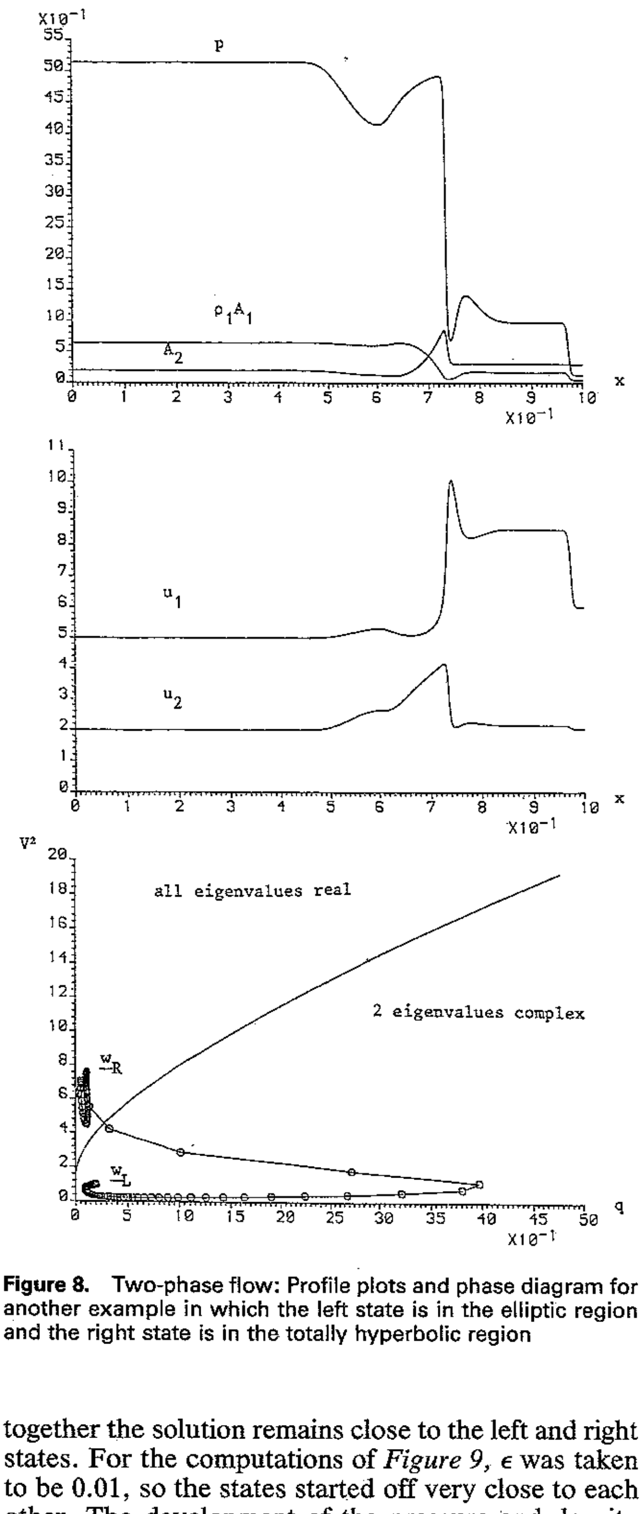

Figure 8. Two-phase flow: Profile plots and phase diagram for another example in which the left state is in the elliptic region and the right state is in the totally hyperbolic region

together the solution remains close to the left and right states. For the computations of *Figure 9*, ϵ was taken to be 0.01, so the states started off very close to each other. The development of the pressure and density profiles is shown, as is the phase diagram at time t = 0.15. The ill-posedness is clear, and a bizarre shock and rarefaction structure is developing and growing in time. There could be no possible physical justification for such results. It is also interesting to note from the phase diagram that one of the portions of the phase curve seems to be trying to plot a path along the q and V^2 axes where the characteristics are real. If the computational mesh is extended farther along the x-axis, then the curious shock system continues to grow in amplitude and develop ever more components. Eventually, after a great length of time (many thousands of time steps) the growth is such that the pressure becomes negative and computations cannot be continued. The phase paths wander ever farther along the q-axis in the phase diagram, and the convolutions become more and more complicated. It is interesting to contrast these results to those obtained for three-phase oil recovery flows[27] in which the elliptic region was closed in space so that such pressure profiles could not grow unboundedly in amplitude.

We have also made calculations similar to those in *Figure 9* but with the left state in the totally hyperbolic region. As expected, the solution remains at all times close to the initial data.

Conclusions and discussion

Having examined the numerical and analytical properties of the two sets of equations discussed above, we are now in a position to comment on the future for the numerical solution of mixed problems. To begin with, we have shown that although the inclusion of some numerical viscosity will always help a numerical scheme to perform a little better when it is applied to a mixed problem, viscosity does not constitute a universal cure to all the problems of mixed systems. In fact in a large number of cases the successful solution of such problems is a somewhat arbitrary affair and requires no little amount of luck to ensure that errors that are likely to lead to exponentially growing solution components are never introduced into the calculations. The situation is especially bad when the system in question is large and complicated so that there is no hope of gaining much analytical information about the components of the solution that we might expect to encounter. On the positive side, we have seen that surprisingly good numerical solutions have been obtained in some cases. We therefore conclude that although numerical computations for mixed systems are not out of the question, extreme care must be taken and all results regarded with caution.

It is possible that in the future, specialized numerical methods will be developed specifically for mixed problems that are able to filter out unwanted parasitic waves. Although such mixed TVD methods could lead to increased reliability and convenience, the underlying ill-posedness of the problem when any of the elliptic regions are entered and the infinite time boundary condition that is then required will always militate against the totally satisfactory numerical solution of mixed conservation law systems.

Also, it has been shown that mixed systems cannot be justified by the simple statement that two states in totally hyperbolic regions are guaranteed never to stray into the nonhyperbolic region. Although it is conceivable that there *are* mixed systems that are "correct" as far as the mathematical modelling is concerned and in which physically sensible initial and boundary data will mean that the nonhyperbolic regions that are pres-

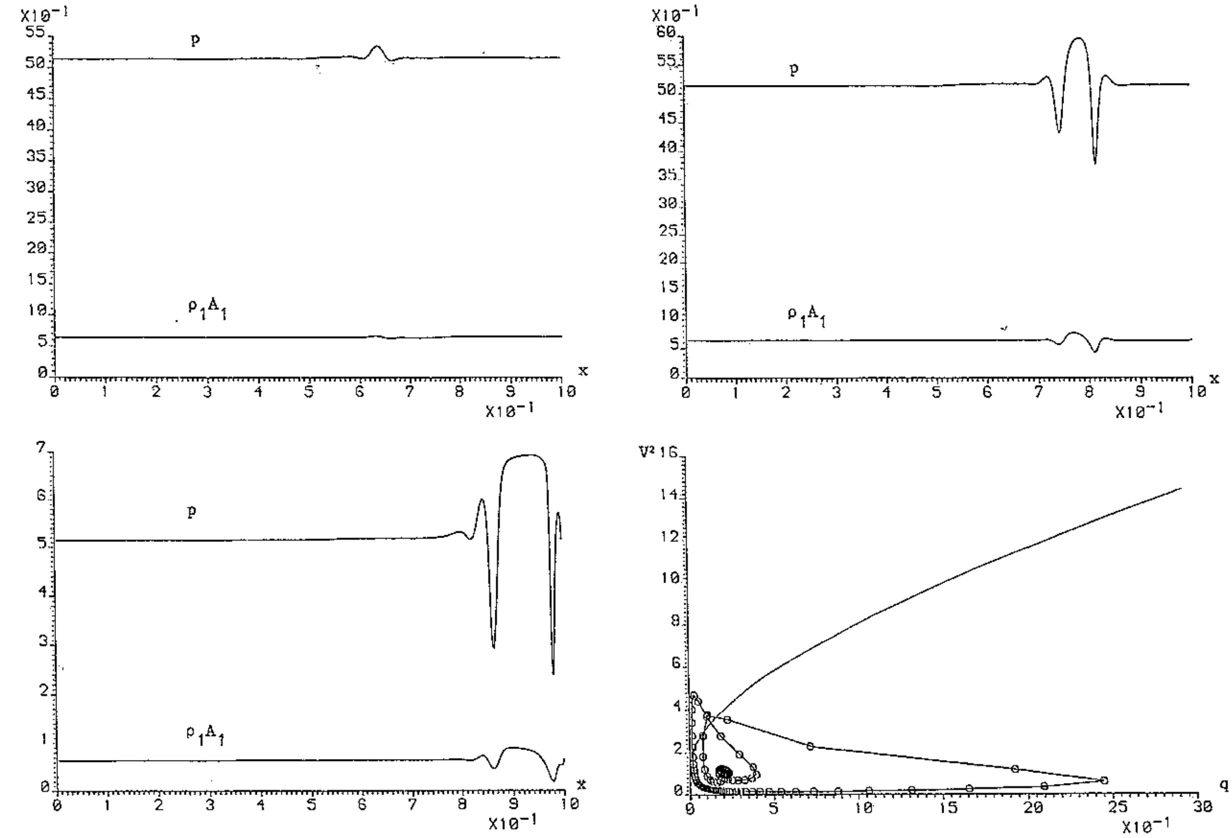

Figure 9. Two-phase flow: "Well-posedness" test in which both states are close to each other in the elliptic region

ent in the theoretical phase space will never in fact be encountered, there are also evidently examples in which this is not true. Taking the two-phase flow equations, for example, it is clear that any physical experiment that we wish to model in which everything is at rest initially leads to phase paths that are doomed to enter the nonhyperbolic region.

Some discussion of the nature of the ill-posedness of the problem is also relevant. By "ill-posed" we normally mean simply that small changes in the initial data can lead to large changes in the solution. Certainly, the linearization of both the sets of conservation laws that we have considered leads to an ill-posed problem when diffusive forces are ignored. When these effects are included, the diffusive terms damp the exponential growth in certain cases, but there remain frequencies where the diffusive terms are inadequate to damp exponential modes.

In some ways, mixed systems are a continuing source of controversy. It seems, however, that the long-term solution to the controversy is probably clear; the physical problem is indicating that better modelling is required, and we should tolerate the existence of elliptic regions only for as long as it takes for the modelling task to be carried out successfully. As far as the less physical prototype systems are concerned, these should be used to increase our knowledge about equations of mixed type so that we are able to distinguish between cases in which the nonhyperbolicity has only a theoretical role to play and those in which the appearance of elliptic regions is a severe fault of the model.

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