The slow spreading of several viscous films over a deep viscous pool

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In this study a previously derived two-dimensional model is used to describe the slow spreading of viscous films on the surface of a quiescent deep viscous pool due to gravity. It is assumed that the densities and viscosities of the fluids in the films and pool are comparable, but may be different. It is also assumed that surface tension effects are negligible. The fluid in the films and in the pool are both modelled using the Stokes flow equations. By exploiting the slenderness of the spreading films, asymptotic techniques are used to analyse the flow. It is shown that the dominant forces controlling the spreading are gravity and the tangential stress induced in the films by the underlying pool. As a consequence the rate of spreading of the films is independent of their viscosity. For the case special of a symmetric configuration of films on the surface of the pool the flow is studied by assuming the solution becomes self-similar and hence the problem is recast in a self-similar co-ordinate system. Stokeslet analysis is then used to derive a singular integral equation for the stresses on the interfaces between the films and the pool. The form of this integral equation depends on the configuration of spreading films that are to be considered. A number of different cases are then studied, namely, a single film, two films and an infinite periodic array of films. Finally some results are derived that apply to a general symmetric configuration of films. It is shown that the profile of a spreading film close to its front (where the film thickness becomes zero) is proportional to $x^{1/4}$. It is also shown that fronts move, and hence, the distance between adjacent fronts increase proportional to $t^{1/3}$.

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I. INTRODUCTION

The spreading of fluid films over fluid pools are examples of problems in the field of gravity currents. Such flows are driven by differences in density and arise in many natural and industrial situations. Gravity currents have been thoroughly studied in fluid mechanics since the 1940's; a good review of the progress that has been made is given in Huppert[22], which describes not only theoretical results but also natural and industrial applications of such flows. Examples of gravity currents that occur in nature include the propagation of air currents, the spreading of oil slicks, saline currents in the ocean, intrusion of clouds in the atmosphere, the spreading of lava, magma flows and the evolution of snow avalanches[4,5,35]. Some industrial examples include accidents in which dense gases are released, the flows in glass furnaces and other areas of glass manufacture including optical fibres[11,13,20].

Some of these physical situations involve the spreading of one fluid on the surface of another. For example, studies have considered the problem of oil spreading over the sea[4,18]. This is an example of a viscous fluid (oil) spreading over the surface of a relatively inviscid fluid (water). In these studies estimates were derived for the length of the spreading oil film as a function of time. Lister *et al.*[26] were the first to consider the problem of a viscous gravity current intruding along an interface between two ambient stably stratified viscous fluids. For the case of deep ambient layers and the case of a shallow lower layer with a fixed lower boundary the nonlinear equations which govern the spreading were derived. Similarity solutions to these governing equations were obtained by assuming that the volume of fluid in the spreading film increased proportional to t^α (with $\alpha \geq 0$). The theoretical predictions on the shape and length of the intruding film were shown to be in good agreement with a series of experiments. More recently, in Koch *et al.*[25] a related problem was considered in which a buoyant viscous drop was allowed to spread below a free fluid surface. Boundary integral methods were used to find numerical solutions for the initial spreading of the drop. To compliment this an asymptotic description of the drop spreading extensively was also given. It was found that different physical mechanisms are dominant in controlling the spreading depending on the viscosity contrasts between the drop and ambient fluid. For very low viscosity drops it was shown that the greatest resistance to spreading occurs at the rim of the drop, for drops with intermediate viscosity the dominant mechanisms controlling the spreading is the shear stress at the drops lower surface. Furthermore, for drops with high viscosity the radial stresses within the drop control the spreading. More recently, in Foster *et al.*[13], a problem akin to that considered in[26] was revisited in the context of glass manufacture. Here, the spreading of a *log* (a film of molten glass foam) over the surface of a pool of molten glass was considered in two dimensions. Analytical solutions were obtained by means of similarity reductions for the case of a fixed volume of fluid in the film. In the current study we extend the previous results to describe the evolution of multiple spreading viscous

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films. This extension is both of theoretical interest, and has applications in the glass making industry where observations suggest that the spreading films may influence one another's motion by driving flows in the underlying viscous pool. Understanding how *logs* drive one anothers motion is of paramount importance when attempting to ensure that bubbles from the glass foam are not frozen into the glass product. For a more in-depth discussion of the applications to glass making we refer the reader to Foster *et al.*[13].

The model assumes that the densities and viscosities of the fluid in the films and pool are comparable, but may be different. It is also assumed that surface tension effects are negligible. The fluid in the films and in the pool are modelled using the Stokes flow equations. Since we are interested in situations where the depth of the pool is much larger than the spatial extent of the spreading films we model the pool as occupying the lower half-plane. The small aspect ratio, ϵ (= typical film depth/typical film length), of the spreading layers is exploited to derive a system of PDEs for the lowest order evolution of the spreading films. For the special case of a symmetric configuration of films it is assumed that the solution becomes self-similar, hence, the flow in the deep pool is analysed by considering the problem in self-similar coordinates. This leads to a singular integral equation that relates the stresses on the interfaces between the spreading layers and the pool. We then study a number of different configurations of films and also derive some results that apply to a general symmetric configuration of films. In each case, once the appropriate integral equation has been solved a simple ODE problem can then be solved to determine the evolution of the films. Hence a description of the spreading of the films is derived and predictions are made concerning the flow induced in the underlying pool.

In the parameter regime under consideration it is found that the dominant forces controlling the spreading are the gravitational force due to the buoyant films and the viscous stress in the pool. As a consequence, the spreading of the films is independent of their viscosity. It is also shown that the films float on the pool at a height determined according to a relation consistent with Archimedes' principle. It is shown that the position of the fronts of each film move proportional to $t^{1/3}$. As a consequence the distance between adjacent fronts also increases proportional to $t^{1/3}$. For a single film, two films and an infinite periodic array of films analytical descriptions of the films evolution are derived. In these cases it is found that the gradient of the profile of the films near their fronts is infinite and close to this front the profile of the film is proportional to $x^{1/4}$. In section VIII it is shown this asymptotic behaviour also holds for a general symmetric configuration of films.

II. PROBLEM FORMULATION

As discussed, a model for the slow spreading of films of incompressible viscous fluid over a quiescent deep incompressible viscous pool is now set up. A schematic diagram of the flow is shown in figure 1. The fluid in the deep pool is referred to as fluid 1, and the fluid in the spreading layers as fluid 2. The density and viscosity of fluids 1 and 2 are defined to be ρ_1, μ_1, ρ_2 and μ_2 respectively, with $\rho_1 > \rho_2$ so that the spreading films are less dense than the underlying pool and are therefore stably stratified on the pool's surface. The regime in which both phases are well represented by slow-viscous flows is considered, and so the Stokes flow equations are used to govern the flow in each phase. The deep pool has a stress free top surface (denoted by $y = f(x,t)$). Note that $f(x,t)$ is only defined for intervals of x in which there is no fluid 2 floating on the surface (i.e. where $g(x,t)$ and $h(x,t)$ are zero), as shown in figure 1. The free surface at the top of fluid 2 (denoted by $y = h(x,t)$) is also modelled as stress free. Across the fluid-fluid surface between fluids 1 and 2 (denoted by $y = -g(x,t)$, so that $g(x,t)$ is the depth of the film below the surface of the pool), stresses and velocities are continuous. Note that by defining the surfaces in this manner the thickness of the floating layers is given by $g(x,t)+h(x,t)$. The surfaces $f(x,t)$, $g(x,t)$ and $h(x,t)$ all evolve as material surfaces and hence satisfy the usual kinematic condition. Since the pool is quiescent the velocities in fluid 1 far from the spreading films vanish. Specification of the initial configuration of the films completes the problem definition.

The flow is non-dimensionalised using the following scalings. In what follows hatted variables represent quantities in the pool and barred variables represent quantities in the films. In the spreading films we set $x = L\bar{x}$, $y = h\bar{y}$, $u = \epsilon^{-1}u_0\bar{u}$, $v = u_0\bar{v}$ and $p = \rho_2 g_r h\bar{p}$. In the pool we set $x = L\hat{x}$, $y = L\hat{y}$, $u = \epsilon^{-1}u_0\hat{u}$, $v = \epsilon^{-1}u_0\hat{v}$ and $p = \epsilon^{-1}\rho_2 g_r h\hat{p}$. Time is scaled by setting $t = \epsilon L u_0^{-1}\hat{t}$. Where $\epsilon = hL^{-1}$ and $u_0 = \rho_2 g_r h^3 \mu_2^{-1} L^{-1}$. We note that although we have chosen to denote the non-dimensional time variable with $\hat{t}$ this scaling is used for the flow in both the films and pool, hence, the hatted notation for this variable does not refer exclusively to the time scale in the pool. With these scalings three non-dimensional parameters are shown to characterise the flow. The first parameter, ϵ (= typical film depth/typical film length), is the aspect ratio of the spreading films. The second, μ ($= \mu_1/\mu_2$), is the ratio of the viscosities of the two fluids, and the third, ρ ($= \rho_1/\rho_2 > 1$), is the ratio of the densities of the two fluids. In the case when the spreading films typical length is much greater than their typical depth the small aspect ratio of the films may be exploited by considering the flow in the limit that $\epsilon \to 0$. In this limit both μ and ρ are taken to be $O(1)$. By making regular asymptotic expansions for the dependant variables in the problem it has been shown[13] that to leading order (in ϵ) the motion of the spreading films is governed by

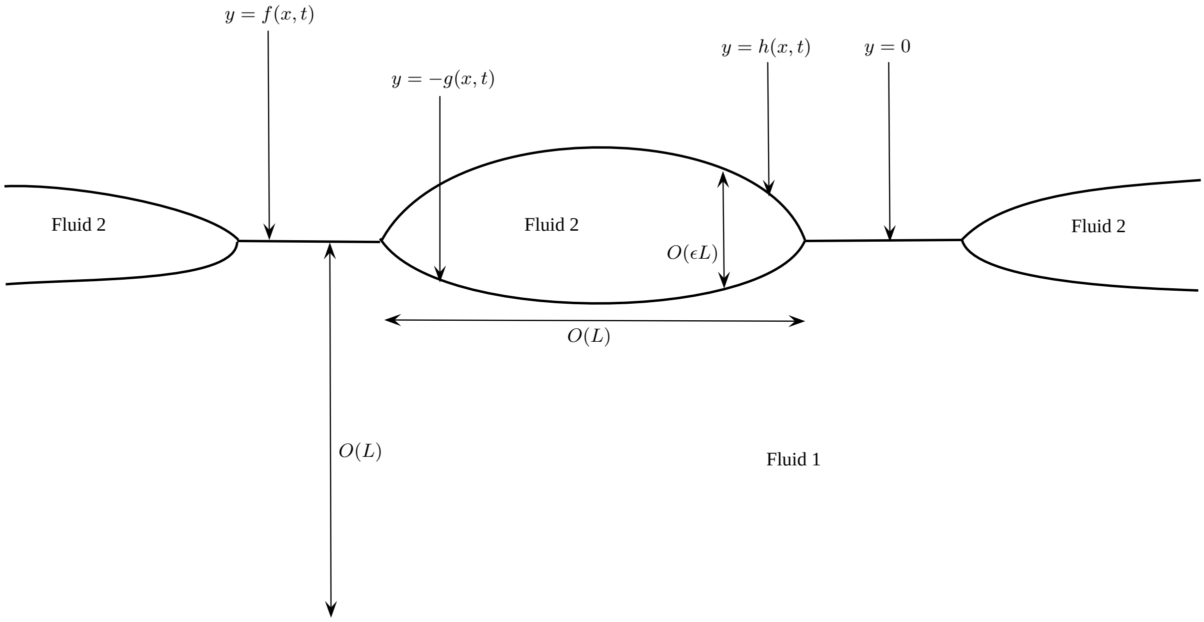

FIG. 1. A schematic diagram of the flow showing characteristics length scales.

the equation of conservation of mass,

$$\frac{\partial}{\partial \hat{t}}(g+h) + \frac{\partial}{\partial \bar{x}}\left(\hat{u}\Big|_{\hat{y}=0}(g+h)\right) = 0. \tag{1}$$

Here t is time, $\bar{x}$ is the horizontal coordinate in the films, $\hat{u}$ is the horizontal component of the fluid velocity in the pool and $\hat{y} = 0$ is the fluid-fluid surface in the scaled co-ordinates. Note that the quantity $g + h$ is the film thickness, this arises since h is defined as the height of the film above the surface of the pool and g is defined as the depth of the film below the surface of the pool. The films also obey a force balance between gravity and the shear stress induced by the underlying pool,

$$(g+h)\frac{\partial h}{\partial \bar{x}} + \mu\frac{\partial \hat{u}}{\partial \hat{y}}\Big|_{\hat{y}=0} = 0, \tag{2}$$

where $\hat{y}$ is the vertical coordinate in the pool. Finally, the films have been shown to float at a height determined by a relation consistent with Archimedes' principle,

$$h = (\rho - 1)g. \tag{3}$$

By re-dimensionalising (1), (2) and (3) it can be seen that the system of equations that governs the spreading of the films is independent of μ_2. Hence, the rate of spreading of the films is independent of their viscosity. This result can be understood intuitively, since the films are slender their viscous stresses impose a negligible contribution to the force balance. In particular, the viscous stresses in the pool dominate those in the films.

III. A SYMMETRIC CONFIGURATION OF FILMS

In order to make further analytical progress with the problem the special case when the configuration of spreading films has reflectional symmetry in the vertical axis is now considered. If the flow has this property then the equations may be reduced to a steady problem by making a self-similar reduction of the form

$$\begin{aligned}\hat{u} &= \hat{t}^{-2/3}U(\phi,\theta),\\ \hat{v} &= \hat{t}^{-2/3}V(\phi,\theta),\\ \hat{p} &= \hat{t}^{-1}P(\phi,\theta),\\ g &= \hat{t}^{-1/3}G(\phi),\\ h &= \hat{t}^{-1/3}H(\phi),\end{aligned} \tag{4}$$

with

$$\phi = \hat{x}\hat{t}^{-1/3} \quad \text{and} \quad \theta = \hat{y}\hat{t}^{-1/3}. \tag{5}$$

Note that by using this reduction some information about the initial configuration of films is inherently given. In order to close the reduced problem is it sufficient to specify the quantity of fluid in each film and one further condition on the location of one of the fronts of each of the films at some reference time (or equivalently the distance between adjacent films at some reference time). It is useful therefore to introduce the notation $\phi = \phi_n$ and $\phi = \phi_{n+1/2}$ as the location of the left and right fronts of the n^{th} film along the line $\theta = 0$. Hence $H(\phi_n) = H(\phi_{n+1/2}) = 0$. We also write $F = \cup_{n=-N}^{n=N}(\phi_n, \phi_{n+1/2})$ so that F is the union of intervals along the line $\theta = 0$ for which H is non-zero, as shown in figure 2. Due to the form of (4) the position of the fronts of the n^{th} film are

$\phi_n = x_n t^{-1/3}$. Hence fronts move proportional to $t^{-1/3}$, and, adjacent fronts separate proportional to $t^{-1/3}$.

Using the reduction (4) and equation (3), (1) and (2) may be written as

$$\frac{d}{d\phi}\left(H\left(U-\frac{1}{3}\phi\right)\right)=0 \tag{6}$$

and

$$\frac{d}{d\phi}\left(\frac{H^2}{2}\right)=\frac{\mu(1-\rho)}{\rho}\frac{\partial U}{\partial \theta} \tag{7}$$

$$\text{on} \quad \theta=0 \quad \text{for} \quad \phi\in F.$$

Since equation (6) holds along the line $\theta = 0$ this may be integrated with respect to ϕ. Owing to the symmetric nature of the flow $U(0,0) = 0$, hence, equation (6) leads to the results,

$$U=\frac{1}{3}\phi \quad \text{on} \quad \theta=0 \quad \text{for} \quad \phi\in F. \tag{8}$$

By introducing a stream function, $\psi(\phi,\theta)$, defined as

$$\frac{\partial\psi}{\partial\theta}=U, \quad -\frac{\partial\psi}{\partial\phi}=V, \tag{9}$$

the problem for the flow in the pool under a general symmetric configuration of spreading films may be written as

$$\nabla^4\psi=0 \tag{10}$$

subject to

$$\frac{\partial^2\psi}{\partial\theta^2}=0 \quad \text{and} \quad \psi=0 \quad \text{on} \quad \theta=0 \quad \text{for} \quad \phi\notin F. \tag{11}$$

$$\frac{\partial\psi}{\partial\theta}=\frac{1}{3}\phi \quad \text{and} \quad \psi=0 \quad \text{on} \quad \theta=0 \quad \text{for} \quad \phi\in F. \tag{12}$$

$$\frac{\partial\psi}{\partial\theta}\to 0\,, \ \frac{\partial\psi}{\partial\phi}\to 0 \quad \text{as} \quad \phi\to\pm\infty. \tag{13}$$

$$\frac{\partial\psi}{\partial\theta}\to 0\,, \ \frac{\partial\psi}{\partial\phi}\to 0 \quad \text{as} \quad \theta\to-\infty. \tag{14}$$

Note that the first condition in (12) is (8) written in terms of the stream function ψ. To solve the problem posed by (10) - (14) Stokeslet analysis is used. Consider the stream function of a Stokeslet oriented in the direction of positive ϕ on the line $\theta = 0$ positioned at the point $\phi = \Phi$,

$$\psi_s(\phi,\theta,\Phi)=-\frac{\theta}{2}\ln\left((\phi-\Phi)^2+\theta^2\right)-1. \tag{15}$$

Provided $\Phi\in F$, ψ_s satisfies equation (10), (11), (13) and (14). Hence, to solve the problem posed by (10) - (14) a superposition of Stokeslets must be picked such that (12) is satisfied. Using equation (12) this requirement may be written as

$$\lim_{\theta\to 0^-}⨍_F s(\Phi)\frac{\partial\psi_s}{\partial\theta}(\phi,\theta,\Phi)\,d\Phi=\frac{1}{3}\phi \quad \text{for} \quad \phi\in F, \tag{16}$$

where s is a function that describes the Stokeslet concentration in F. Note that this equation cannot simply be evaluated on $\theta = 0$ due to the singular nature of (15), instead the limit that $\theta\to 0^-$ must be taken. Note also that the 'dashed' integral sign indicates that the integral is to be understood in the Cauchy Principal value sense. Taking the derivative of (16) with respect to ϕ and taking the limit that $\theta\to 0^-$, equation (16) may be written as

$$⨍_F \frac{s(\Phi)}{\Phi-\phi}d\Phi=\frac{1}{3} \quad \text{for} \quad \phi\in F. \tag{17}$$

The solution to equation (17) depends critically on the form of the domain of integration F. In sections IV, V, VI and VII the function s is computed for a number of different examples. In the interests of brevity and clarity of the problem structure we continue to solve the problem assuming that s may be determined from equation (17).

The tangential stress on the fluid-fluid surface can be found by noting that the Stokeslet solution, ψ_s, is the flow generated by a stress singularity of strength 2π. Hence

$$\left.\frac{\partial U}{\partial\theta}\right|_{\theta=0}=2\pi s(\phi) \quad \text{for} \quad \phi\in F. \tag{18}$$

Substitution of equation (18) into (7) and integrating with respect to ϕ gives

$$H^2=4\pi\frac{\mu(1-\rho)}{\rho}\int_F s(\phi)d\phi \quad \text{for} \quad \phi\in F \tag{19}$$

as an expression for H^2. A number of examples of different configurations are studied in the following sections.

IV. A SINGLE SPREADING FILM

In this section some of the results of Foster *et al.*[13] and Lister *et al.*[26] are rederived. For the case of a single spreading film F takes a particularly simple form, a single section of the line $\theta = 0$. Due to the assumed symmetric nature of the problem, in this case, $\phi_1 = -\phi_{3/2}$. Hence, equation (17) takes the form

$$\frac{1}{3}=⨍_{-\phi_1}^{+\phi_1}\frac{s(\Phi)}{\Phi-\phi}d\Phi \quad \text{for} \quad -\phi_1<\phi<+\phi_1. \tag{20}$$

Equation (20) may be solved using standard results, see Muskhelishvili[30], to give

$$s(\phi)=\frac{1}{3\pi}\frac{\phi}{\sqrt{\phi_1^2-\phi^2}} \quad \text{for} \quad -\phi_1<\phi<+\phi_1. \tag{21}$$

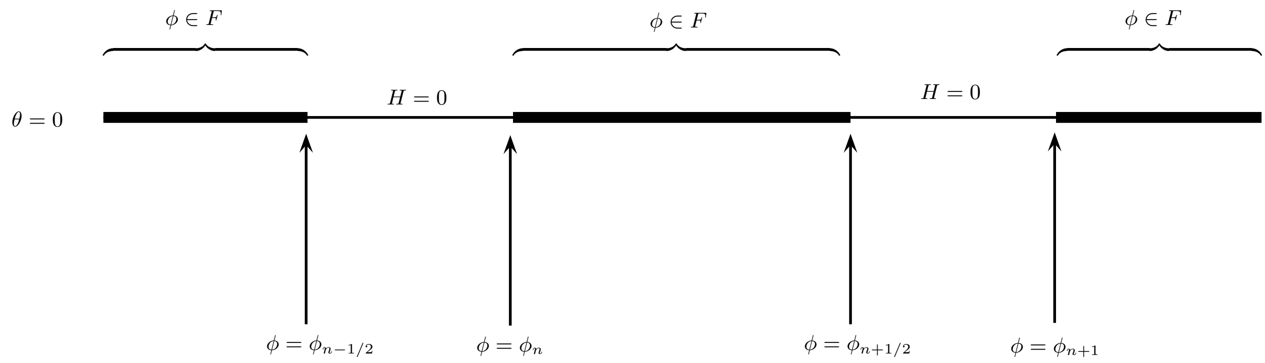

FIG. 2. A schematic of the problem in the self-similar coordinate system.

Following the steps outlined between equations (17) and (19), and, imposing the conditions $H(-\phi_1) = H(+\phi_1) = 0$, the profile of the film can be shown to take the form

$$H^2 = \frac{4}{3}\frac{\mu(\rho-1)}{\rho}\sqrt{\phi_1^2 - \phi^2} \quad \text{for} \quad -\phi_1 < \phi < +\phi_1. \tag{22}$$

Closure of the problem, finding ϕ_1, is completed by specifying the total amount of fluid in the film. This is imposed by consideration of the equation

$$\int_{-\phi_1}^{+\phi_1} H d\phi = M_1. \tag{23}$$

Where M_1 is the quantity of fluid in the film.

For the purposes of demonstration we use the example of $M_1 = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$. In this case it can be shown numerically that $\phi_1 \approx 0.5481$. The predicted profile for H and streamlines of the flow generated in the pool are shown in figures 3 and 4.

V. A FINITE NUMBER OF SYMMETRIC FILMS

In this section the problem for a finite number of symmetrically arranged films is studied. The solutions that we derive demonstrate how films can influence one anothers motion. This is an interesting extension of previous results and has applications in physical situations (e.g. glass manufacture[13]). In this case equation (17) may be written as

$$\frac{1}{3} = \fint_{F^-} \frac{s(\Phi)}{\Phi - \phi} d\Phi + \fint_{F^+} \frac{s(\Phi)}{\Phi - \phi} d\Phi \quad \text{for} \quad \phi \in F. \tag{24}$$

Where $F^+ = \{\phi \in F | \phi > 0\}$ and $F^- = \{\phi \in F | \phi < 0\}$. The symmetry of the problem may then be exploited by substituting $\Phi = -\Phi$ in the first term on the RHS of (24). Hence

$$\frac{1}{3} = \fint_{F^+} s(\Phi)\frac{2\Phi}{\Phi^2 - \phi^2} d\Phi \quad \text{for} \quad \phi \in F^+. \tag{25}$$

To make further analytical progress we put $\Phi^2 = q$ and $\phi^2 = p$, and also define $\phi_n^2 = p_n$, $s_1(q) = s(\Phi)$ and $\bar{F} = \cup_{n=1}^{n=N}(p_n, p_{n+1/2})$. Equation (25) may then be written as

$$\frac{1}{3} = \fint_{\bar{F}} \frac{s_1(q)}{q - p} dq \quad \text{for} \quad p \in \bar{F}. \tag{26}$$

Equation (26) takes the form of a Fredholm singular integral equation of the first kind, where the domain of integration, $\bar{F}$, is a union of disjoint sections of the real line. Equation (26) can be inverted for s_1 using results in Muskhelishvili[30] to give

$$s_1(p) = -\frac{1}{3\pi^2}\prod_{n=1}^{n=N}\frac{1}{\sqrt{-(p_n - p)(p_{n+1/2} - p)}}\left(\fint_{\bar{F}} \frac{1}{q - p}\prod_{n=1}^{n=N}\sqrt{-(p_n - q)(p_{n+1/2} - q)}dq + c\right). \quad \text{for} \quad p \in \bar{F}. \tag{27}$$

Where c is a constant to be determined. Calculating the integral in equation (27) is, in general, a problem that must be treated numerically. However, some analyt-

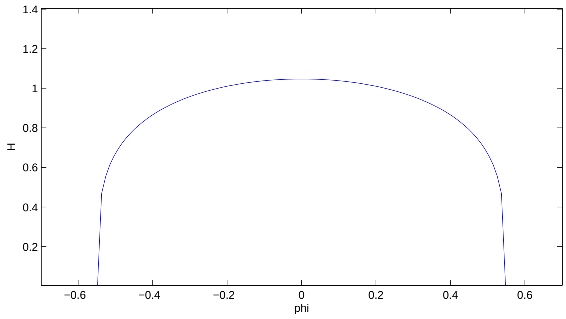

FIG. 3. $H(\phi)$ for $M_1 = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$.

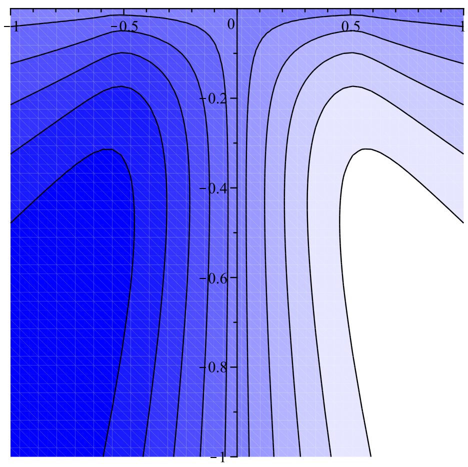

FIG. 4. The streamlines of the flow in the pool in the (ϕ, θ) plane for $M_1 = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$. These streamlines have been computed numerically integrating the product of s and ψ_s on the domain F and plotting the contours of the resulting function.

ical progress can be made by simplifying to some special cases. Having made analytical progress with the problem of an isolated film we now consider multiple films. In the next section we will show that substantial analytical progress can be made with the dual film problem. For larger numbers of films some analytic progress can be made, however, these problems involve elliptic functions (amongst others) that require numerical treatment and in keeping with the spirit of this paper we do not consider them here.

VI. TWO SYMMETRIC FILMS

For the case of two films the integral in equation (27) can be calculated analytically by standard methods, see Abramowitz[1]. In terms of the ϕ coordinate system it may be written that

$$s(\phi) = \frac{1}{3\pi}\frac{\phi^2 + c_1}{\sqrt{(\phi^2 - \phi_1^2)(\phi_{3/2}^2 - \phi^2)}}$$
$$\text{for} \quad \phi \in (-\phi_{3/2}, -\phi_1) \cup (\phi_1, \phi_{3/2}). \tag{28}$$

Where c_1 is a constant that may be determined by requiring that $H(\pm\phi_1) = H(\pm\phi_{3/2}) = 0$. Hence, $c_1 = -((\phi_{3/2} - \phi_1)/2)^2$. Equation (19) becomes

$$H^2 = \frac{4}{3}\frac{\mu(1-\rho)}{\rho}\int \frac{\phi^2 - ((\phi_1 + \phi_{3/2})/2)^2}{\sqrt{(\phi^2 - \phi_1^2)(\phi_{3/2}^2 - \phi^2)}}d\phi$$
$$\text{for} \quad \phi \in (-\phi_{3/2}, -\phi_1) \cup (\phi_1, \phi_{3/2}). \tag{29}$$

The integral in equation (29) must be computed numerically. However, some manipulations can be made to minimize the amount of computation. Since the film thickness is zero at the points $\phi = \pm\phi_1$ and $\phi = \pm\phi_{3/2}$ equation (29) may be be written as

$$H^2 = \frac{4}{3}\frac{\mu(\rho-1)}{\rho}\int_{\phi}^{\phi_{3/2}} \frac{\phi^2 - ((\phi_1 + \phi_{3/2})/2)^2}{\sqrt{(\phi^2 - \phi_1)(\phi_{3/2}^2 - \phi^2)}}d\phi$$
$$\text{for} \quad \phi \in (-\phi_{3/2}, -\phi_1) \cup (\phi_1, \phi_{3/2}). \tag{30}$$

The problem is then reduced to finding ϕ_1 and $\phi_{3/2}$. One condition on one of the fronts of each of the films must be given. In this example we choose to specify the value of ϕ_1, since this is the shortest distance between films at $\hat{t} = 1$. A second condition, to find $\phi_{3/2}$ is to specify the total amount of fluid in each film. In the same way as section IV the equation

$$\int_{\phi_1}^{\phi_{3/2}} H d\phi = M_2 \tag{31}$$

is used. To determine $\phi_{3/2}$ from equations (30) and (31) a shooting method is employed using the following steps.

- A guess for $\phi_{3/2}$ is made.
- The RHS of (30) is then computed numerically using an adaptive Gauss-Kronrod quadrature[34] at a number of equally spaced values of ϕ on the interval $(\phi_1, \phi_{3/2})$.
- From this numerical approximation to H a value for the LHS of (31) is computed using a composite Simpson's rule. If the LHS of (31) is less than M_2 the guess for $\phi_{3/2}$ is increased. However if the LHS of (31) is greater than M_2 the guess for $\phi_{3/2}$ is decreased (since the LHS of (31) is a monotonic increasing function of $\phi_{3/2}$).
- This process is repeated until the LHS of (31) is equal to $M_2 \pm E$, where E is some error tolerance.

For the purposes of demonstration we choose to use the example of choosing $\phi_1 = 1$, $M_2 = 1$ and

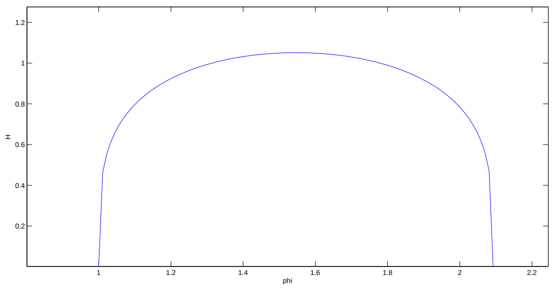

FIG. 5. $H(\phi)$ for for $M_2 = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$.

$2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$. Using 100 equally spaced values of ϕ on the interval $(\phi_1, \phi_{3/2})$ and a value of $E = \pm 10^{-3}$ it is found that $\phi_{3/2} \approx 2.0924$, the predicted profile for H and streamlines of the flow generated in the pool are shown in figures 5 and 6. It is at this stage that a comparison can be made between the single film and dual film problems. We note that in the single film case (with a unit mass and $\mu(\rho-1)/(3\rho) = 1$) the length of the film was predicted to be approximately $2 \times 0.5481 = 1.0962$. As we have just seen, our model predicts that two films (each with a unit mass and $\mu(\rho-1)/(3\rho) = 1$) each have a length of approximately $2.0924 - 1 = 1.0924$. In addition, the profiles predicted in the single and dual film cases are not dissimilar (see figures 3 and 5). This is an interesting result that indicates that the profile of a film is not significantly affected by introducing a neighbouring film. However, although the profile is not appreciably changed the motion of the film is qualitatively altered by the presence of a neighbouring film, in that, it is pushed along the surface of the pool by the flow generated by its neighbour.

VII. AN INFINITE PERIODIC ARRAY OF FILMS

In this section a problem is considered for an infinite periodic array of spreading films. This problem gives rise to some novel mathematical results and could be thought of as a limiting case for a large (but finite) array of films. We anticipate that the infinite periodic array problem gives the lowest order approximation for the evolution of the films sufficiently far from the edges of a large finite array. In other words, we expect the solutions that we derive to approximately describe the evolution of films near to the centre of a large finite array. As we shall see later this approach has the advantage that the approximate solution may be found analytically, whereas, equation (27) would require non-trivial numerical treatment to find solutions. At this stage we should point out that there are some difficulties associated with studying the infinite periodic array problem. For example, it predicts infinite velocities in the far-field (which are, of course,

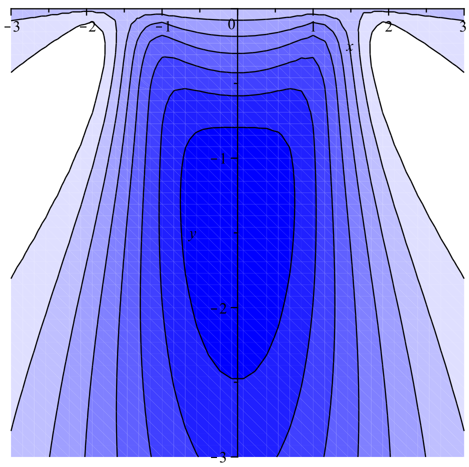

FIG. 6. The streamlines of the flow in the pool in the (ϕ, θ) plane for $M_2 = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$. These streamlines have been computed numerically integrating the product of s and ψ_s on the domain F and plotting the contours of the resulting function.

non-physical) however we proceed to solve the problem bearing in mind that we can regard the solution as an approximation to the large finite array problem.

We denote the distance between the centre of adjacent films at time $\hat{t} = 1$, D, and the length of each film at time $\hat{t} = 1$, $2a$. In the interests of algebraic clarity the horizontal co-ordinate is scaled with D by putting $\phi = D\bar{\phi}$. Then the distance between the centre of adjacent films in the $\bar{\phi}$ coordinate system is one, and the extent of each film is $2a/D = 2\bar{a}$. Without loss of generality we choose to position the centres of the films at $\bar{\phi} = n + 1/2$ along the line $\theta = 0$, where $n \in \mathbb{Z}$. Equation (17) may be written as

$$\frac{1}{3} = \sum_{n=-\infty}^{n=+\infty} \int_{(n+1/2)-\bar{a}}^{(n+1/2)+\bar{a}} \frac{s(\Phi)}{\Phi - \bar{\phi}} d\Phi \quad \text{for} \quad \bar{\phi} \in (n + 1/2 - \bar{a}, n + 1/2 + \bar{a}). \tag{32}$$

We now put $\Phi = n + 1/2 + q$ and $\bar{\phi} = m + 1/2 + p$ with $p, q \in (-\bar{a}, +\bar{a})$ and $m \in \mathbb{Z}$. Owing to the periodicity of the problem the function s has the property that $s(z) = s(z + k)$ for any $k \in \mathbb{Z}$. Therefore, the problem may be reduced to considering a single period of the function s by writing equation (32) as

$$\frac{1}{3} = \int_{-\bar{a}}^{+\bar{a}} s(q) \sum_{n=-\infty}^{n=+\infty} \frac{1}{(n+q) - (m+p)} dq \quad \text{for} \quad p \in (-\bar{a}, +\bar{a}). \tag{33}$$

This is a singular Fredholm integral equation of the first kind for the function s. The form of the kernel in equation (33) may be manipulated into a simpler form by noting that the kernel is related to the Hurwitz-Zeta function[3]. This function's properties have been considered previously, and it can be shown using standard techniques[24,37] that

$$\sum_{n=-\infty}^{n=+\infty} \frac{1}{(n+q)-(m+p)} = \pi \cot \pi (q-p). \tag{34}$$

Hence equation (33) may be written as

$$\frac{1}{3} = \pi \fint_{-\bar{a}}^{+\bar{a}} s(q) \cot \pi (q-p) dq$$
$$\text{with} \quad p \in (-\bar{a}, +\bar{a}). \tag{35}$$

In order to progress with finding a solution to equation (35) we follow the ideas in Muskhelishvili[30] and project (35) into the complex plane by making the substitutions $Q = \exp(2\pi i q)$, $P = \exp(2\pi i p)$, $S(Q) = s(q)$ and $\alpha_0 = \exp(2\pi i \bar{a})$. Then equation (35) may be written as

$$\frac{1}{3} = \fint_A \frac{S(Q)}{Q-P} dQ \quad \text{for} \quad P \in A. \tag{36}$$

Here A is the part of the unit circle in the complex plane connecting α_0^* and α_0, where α_0^* is the complex conjugate of α_0. Equation (36) may be inverted using results in Estrada *et al.*[10] to give

$$S(P) = \frac{1}{3\pi} \frac{P + c_2}{\sqrt{(P-\alpha_0)(\alpha_0^* - P)}} \quad \text{for} \quad P \in A. \tag{37}$$

Where c_2 is an undetermined complex constant. So that the function s is purely real, and so that it is anti-symmetric on the domain $(-\bar{a}, +\bar{a})$, c_2 is set equal to minus one. This allows the function s to be written in terms of p as

$$s(p) = \frac{1}{3\pi} \frac{\sin(\pi p)}{\sqrt{\sin(\pi(a+p))\sin(\pi(\bar{a}-p))}}. \tag{38}$$

Using equations (19) and (38) and following the steps outlined in section III gives

$$H^2 = \frac{4}{3}\frac{\mu(1-\rho)}{\rho} \int_{-\bar{a}}^{+\bar{a}} \frac{\sin(\pi p)}{\sqrt{\sin(\pi(\bar{a}+p))\sin(\pi(\bar{a}-p))}} dp. \tag{39}$$

The integral in equation (39) may be calculated and imposing that $H(-p) = H(+p) = 0$ gives

$$H^2 = \frac{4}{3}\frac{\mu(\rho-1)}{\rho} \ln\left(\frac{\cos(\pi p) + \sqrt{\cos^2(\pi p) - \cos^2(\pi \bar{a})}}{\cos(\pi \bar{a})}\right). \tag{40}$$

In order to close the problem a value for $\bar{a}$ must be found. In the same way as sections IV and VI the following condition on the total amount of fluid in each film is imposed

$$\int_{-\bar{a}}^{+\bar{a}} H dp = M_\infty. \tag{41}$$

For the purposes of demonstration we choose to use the example of $M_\infty = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$. In this case

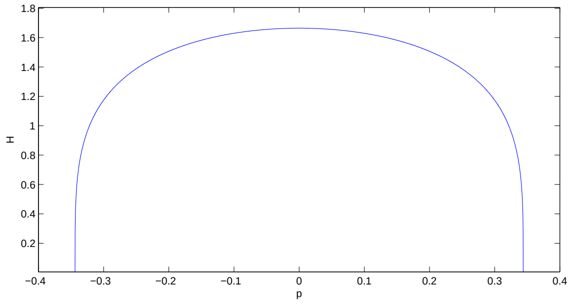

FIG. 7. $H(p)$ for $M_\infty = 1$ and $2\frac{\mu}{3}\frac{\rho-1}{\rho} = 1$.

it can be shown numerically that $\bar{a} \approx 0.3438$. The predicted profile for H is shown in figure 7.

Note that it can be systematically shown, by expanding equation (35) for small $\bar{a}$, that the problem studied in this section reduces to the problem for a single spreading film. This corresponds to films being very well separated and are therefore not significantly affected by the flow generated by neighbouring films.

VIII. BEHAVIOUR AT THE FRONT

In sections IV, VI and VII, the problems of a single spreading film, two spreading films and an infinite periodic array of spreading films have been studied. We note that a challenge inherent in dealing with more general configurations of films comes when computing the more complex counterparts of the integrals derived from equations (17) and (19). Many of the integrands exhibit singular behaviour near at the endpoints of the integral and due to the absence of analytical solutions, the integrals may have to be computed numerically. It is therefore useful to understand the nature of these singularities analytically.

For the case of a single spreading film it is straight forward to determine these behaviours. Examination of equation (21) shows

$$s \sim K_0 \left(\phi_1 + \phi\right)^{-1/2} \quad \text{as} \quad \phi \to -\phi_1^+, \tag{42}$$

$$s \sim K_1 \left(\phi_1 - \phi\right)^{-1/2} \quad \text{as} \quad \phi \to +\phi_1^-. \tag{43}$$

Furthermore, direct inspection of equation (22) shows

$$H \sim K_2 \left(\phi_1 + \phi\right)^{1/4} \quad \text{as} \quad \phi \to -\phi_1^+, \tag{44}$$

$$H \sim K_3 \left(\phi_1 - \phi\right)^{1/4} \quad \text{as} \quad \phi \to +\phi_1^-. \tag{45}$$

Here K_i are constants (which, in this case, could be found from the exact solutions (21) and (22)). For the case of two spreading films equations (28) and (30) reveal that the asymptotic behaviours of both s and H also take the forms (42) - (45). This can also be shown to be true for an infinite periodic array of films by expanding trigonometric terms in equation (38) and (40).

In light of the fact that all the cases examined thus far appear to have a generic behaviour near the fronts of each film one might anticipate that this is the case for any spreading film. In order to support this claim we pose the following generalised problem.

Equation (17) is a relationship that has been derived by seeking a distribution of Stokeslets that will satisfy a given velocity condition along the fluid-fluid surface, see section III. Thus far, in the current study the velocity condition to be satisfied has been $U(\phi,0)=\phi/3$ for $\phi\in F$. To answer the aforementioned question we introduce a general, but well behaved, function $U_G(\phi)$ (with the property that $U_G(0)=0$) and assume that the velocity condition to be satisfied is $U(\phi,0)=U_G(\phi)$ for $\phi\in F$. We also introduce a film that has a front at $\phi=\phi_G$ and lies in $\phi>\phi_G$. Following the working in section III and carrying out an analysis local to $\phi=\phi_G$ an equation that takes the form

$$\frac{dU_G}{d\phi}=\fint_{\phi_G}^{\infty}\frac{s(\Phi)}{\Phi-\phi}d\Phi \quad \text{with} \quad \phi\in(\phi_G,+\infty), \tag{46}$$

is derived. In order to understand the behaviour of the film near its front the behaviour of s as $\phi\to\phi_G^+$ must first be determined. By introducing a constant δ, such that $\delta\ll 1$ but $\phi<\phi_G+\delta$ equation (46) may be written as

$$\frac{dU_G}{d\phi}=\fint_{\phi_G}^{\phi_G+\delta}\frac{s(\Phi)}{\Phi-\phi}d\Phi+\int_{\phi_G+\delta}^{\infty}\frac{s(\Phi)}{\Phi-\phi}d\Phi. \tag{47}$$

Assuming that s blows-up but is still integrable as $\phi\to\phi_G^+$ motivates expanding $s\sim K(\phi-\phi_G)^{-p}$ as $\phi\to\phi_G^+$ with $0<p<1$ and K a constant. This gives

$$\begin{aligned}\frac{dU_G}{d\phi}=&\fint_{\phi_G}^{\infty}\frac{K(\Phi-\phi_G)^{-p}}{\Phi-\phi}d\Phi\\&+\int_{\phi_G+\delta}^{\infty}\frac{f(\Phi)-K(\Phi-\phi_G)^{-p}}{\Phi-\phi}d\Phi.\end{aligned} \tag{48}$$

Substituting $z-\phi_G=\phi u$ in the first term on the RHS of equation (48) gives

$$\begin{aligned}\frac{dU_G}{d\phi}=&-K\phi^{-p}\fint_{0}^{\infty}\frac{u^{-p}}{1-u}du\\&+\int_{\phi_G+\delta}^{\infty}\frac{f(\Phi)-K(\Phi-\phi_G)^{-p}}{\Phi-\phi}dz.\end{aligned} \tag{49}$$

Recalling the result

$$\fint_{0}^{\infty}\frac{u^{-p}}{1-u}du=-\pi\cot(\pi p), \tag{50}$$

allows equation (49) to written as

$$\begin{aligned}\frac{dU_G}{d\phi}=&K(\phi-\phi_G)^{-p}\pi\cot(\pi p)\\&+\int_{\delta}^{\infty}\frac{f(\Phi)-K(\Phi-\phi_G)^{-p}}{\Phi-\phi}d\Phi.\end{aligned} \tag{51}$$

To determine the asymptotic behaviour of s as $\phi\to\phi_G^+$ the terms in equation (51) are examined. By assumption, the term on the LHS is bounded. Assuming that s is well behaved on the domain $(\phi_G,+\infty)$, a physically reasonable expectation, the second term on the RHS converges, and so it too is bounded. It follows that the first term on the RHS must also be bounded as $\phi\to\phi_G^+$. In order for this to be true the exponent $p=1/2$, since $0<p<1$ and $\cot\pi/2=0$. Hence

$$s\sim K_4(\phi-\phi_G)^{-1/2} \quad \text{as} \quad \phi\to\phi_G^+, \tag{52}$$

and using equation (19) it can be seen that

$$H\sim K_5(\phi-\phi_G)^{1/4} \quad \text{as} \quad \phi\to\phi_G^+, \tag{53}$$

where K_i are constants. Hence, provided that the velocity along the fluid-fluid surface is well behaved, the profile of any spreading film close to its fronts has the form (53). We note that the singular behaviour exhibited by equation (52) has been reported on previously by Lister *et al.*[26] and by Moffatt[28]. The structure of the singularity is also closely related to that seen in the problem for flow past an ellipsoid as considered by Hinch[16] and Jeffery[23].

IX. FLOW IN THE FAR FIELD

In this section we discuss some of the aspects of the far field flow generated by the spreading of several symmetrically arranged viscous films. In section III it was shown that the flow in the pool can be described by a symmetric distribution of Stokeslets along the pool's surface (i.e. along $\theta=0$). Therefore, far from the spreading films, the flow in the pool can be approximated by the flow generated by a dipole singularity of strength, B, say. Of course, the effective dipole strength B depends on properties of the spreading films. These properties are the film's mass (or equivalently their length) and the position of each film. In principal it is possible to determine the value of B by solving equation (17) and then computing

$$B=\int_{F^+}s(\phi)d\phi. \tag{54}$$

In the special cases of a single spreading film and two spreading films this process is straight-forward. Using the result (21) the effective dipole strength for a single spreading film of length $2\phi_1$ is given by

$$\frac{\phi_1}{3\pi}. \tag{55}$$

Using the result (28) the effective dipole strength for two spreading films separated by a distance $2\phi_1$ and each with length $\phi_{3/2} - \phi_1$ is given by

$$\frac{1}{12\pi\phi_{3/2}}\left(4{\phi_{3/2}}^2\,\text{EllipticE}\left(\frac{\sqrt{{\phi_{3/2}}^2-\phi_1^2}}{\phi_{3/2}}\right)\right.$$
$$\left.-(\phi_{3/2}-\phi_1)^2\,\text{EllipticK}\left(\frac{\sqrt{{\phi_{3/2}}^2-\phi_1^2}}{\phi_{3/2}}\right)\right). \quad (56)$$

However, in more complicated examples (where there are more films), equation (17) has solution (27) which cannot readily be used in equation (54) to compute B. In these cases B could be computed numerically. Alternatively, it might be possible to find the quantity (54) without finding $s(\phi)$ explicitly, however whether this is possible remains an open question.

X. CONCLUSIONS

A systematic derivation has been given for the equations that govern the spreading of several films of viscous fluid on the surface of a deep pool of more dense viscous fluid. In the parameter regime of interest it has been shown that the dominant force balance is between the gravitational force due to the buoyant layers and the shear stress induced on the films by the viscous stress of the deep pool. As a consequence, the resulting expression for the evolution of the spreading films is independent of their viscosity, and only depends on the density of the films and the density and viscosity of the pool on which they float. It has also been shown that the way in which the films float on the surface of the pool is determined by a relationship that is consistent with Archimedes' principle.

For the special case of a symmetric arrangement of films it has been predicted that the position of any front moves proportional to $t^{1/3}$ and hence the speed of a front is proportional to $t^{-2/3}$. Furthermore, this means that any two adjacent fronts will separate with a distance proportional to $t^{1/3}$. For a single spreading film, two spreading films and an infinite periodic array of films analytical descriptions of the films evolution are derived ((22), (30) and (40)) as well as numerical description of flow generated in the underlying pool. It has also been shown that the model predicts that the gradient of the profile of any fluid film near its front is infinite and close to this front the profile is proportional to $x^{1/4}$.

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