Asymptotic and Numerical Aspects of a Nonlinear Singular Integro-Differential Equation for Dryout in a LMFBR Boiler Tube

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Summary. We examine specific asymptotic and numerical aspects of a mathematical model for determining the dryout point in an LMFBR. By considering a paradigm problem we show that regularisation is essential for the calculation of accurate numerical solutions.

1 Introduction

A typical modern nuclear reactor consists of two key components: a fuel element and a boiling heat exchange component. Nuclear fission and energy conversion take place in the former, where the heat generated is transferred to a coolant. Heat is then transferred from the coolant to water in the boiling component where steam is produced to drive turbines that generate electricity.

A key factor that differentiates between various reactor designs is the type of coolant used. In a Liquid Metal Fast Breeder Reactor (LMFBR), a liquid metal (sodium is a popular choice) is used as the coolant. Such reactors can operate virtually unpressurised, which has the advantage that in the event of a Loss Of Coolant Accident (LOCA) the spread of contaminated material is minimised.

Bundles of steam generating pipes form the boiling component of a LMFBR. Water pumped through the pipes is heated by the counter-current flowing liquid metal in the outer pipe casing After entering as a liquid, the water starts to vapourise, forming a two-phase mixture of water and steam. This gives rise to a range of two-phase flow regimes. In this study we wish to consider the annular flow regime that is established directly before the water turns entirely to steam Annular flow is the predominant regime in a LMFBR pipe and is characterised by a relatively slowly-moving continuous liquid film surrounding a core of fast-flowing gas. The mass transfer in this region is dominated by evaporation of the thin liquid film at its free surface.

The annular flow region terminates at the so-called "dryout point", where complete evaporation of the liquid film occurs. At the dryout point the pipe wall temperature increases sharply since the thermal conductivity of the gas phase is much less than that of the liquid phase. If dryout and rewetting occur periodically, thermal stresses may be set up in the wall which could lead to cracking of the pipe. A good understanding of the mechanics of dryout and the location of the dryout point is therefore essential if one wishes to predict the lifetime of steam generating pipes.

2 Mathematical Model

Space permits only the briefest of descriptions of the mathematical model for the dryout point (for fuller details see [Mph00]). The assumptions inherent in the model include (i) that the flow is two-dimensional and steady, and lubrication theory is valid in the liquid layer (ii) that the gas Reynolds' number is large (iii) that the wall temperature is constant, the liquid is superheated and the mass transfer is determined by a Stefan-type problem and (iv) that the interaction between the fast-flowing gas core and the wall layer may be described using classical thin aerofoil theory. All of these assumptions may be justified using typical LMFBR data. The final result is the (non-dimensional) NLSIDE (nonlinear singular integro-differential equation) valid for $0 \le \bar{x} \le 1$:

$$\left(\bar{\theta}\frac{\bar{h}^3}{3}\left(\frac{1}{\pi}\int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi-\bar{x}}d\xi\right)_{\bar{x}} - \frac{\bar{h}^2}{2}\bar{\tau}\right)_{\bar{x}} = -\frac{\bar{\eta}}{\bar{h}} \qquad (1)$$

Here subscripts denote differentiation, non-dimensional quantities are written with an overbar, and the bar through the integral denotes a Cauchy principal value. The quantity $\bar{\theta} = \epsilon^3 L \rho_\infty U_\infty^2/(\mu U) \sim 1.5$ characterises the relative importance of the pressure variations in the gas core to those in the liquid layer. $\bar{h}(\bar{x})$ denotes the fluid layer height, $\bar{\tau}$ is related to the shear stress exerted by the gas layer on the liquid and $\bar{\eta}$ characterises the strength of the evaporation. The small parameter ϵ denotes the aspect ratio of the fluid layer, ρ_∞ and U_∞ are respectively the density and speed of the free stream, L is the (unknown) length to the dryout point and μ and U are respectively the dynamic viscosity and typical speed of the fluid layer.

Boundary conditions are required for (1). Proceeding on the basis that an nth order NLSIDE normally requires $n+1$ boundary conditions (one for each order and an "inversion" condition) we assert that

$$\bar{h}(0) = 1, \quad \bar{h}(1) = 0, \quad \bar{h}'(0) = 0 \qquad (2)$$

and, at $\bar{x} = 1$,

$$\bar{\theta}\frac{\bar{h}^3}{3}\left(\frac{1}{\pi}\int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi-\bar{x}}d\xi\right)_{\bar{x}} - \frac{\bar{h}^2}{2}\bar{\tau} = 0. \qquad (3)$$

The first two of these conditions reflect the geometry of the problem, and the third insists that the pressure is finite at the onset of annular flow. The final condition (3) expresses the fact that the mass flux from the liquid film must be zero at the dryout point. One further boundary condition is required to determine the length L to dryout; we shall assume here that the pressure p_{g0} is known at the onset of annular flow $\bar{x} = 0$ (a condition that is equivalent to prescribing the total mass flux in the pipe). Thus (in dimensional variables)

$$p_{g0} = p_\infty + \frac{\epsilon \rho_\infty U_\infty^2}{\pi} \int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi} d\xi. \tag{4}$$

3 Paradigm Problem

The NLSIDE (1) with the boundary conditions described above presents an extremely difficult numerical and asymptotic challenge. For the remainder of this study, therefore, we will not consider (1), but instead concentrate our attention on a paradigm problem constructed to allow some simple analysis to be carried out. Specifically, we shall make the (physically untenable) assumptions that $\bar{\tau} \sim 2\bar{\theta}\tau_0^* \bar{x} \bar{h}^{-2}$ and $\bar{\eta} \sim \eta_0^* \bar{h} \bar{\theta}$ and ignore the $\bar{h}^3/3$ term multiplying the singular integral term in (1). The problem then becomes

$$\left(\left(\frac{1}{\pi} \int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi - \bar{x}} d\xi \right)_{\bar{x}} - \tau_0^* \bar{x} \right)_{\bar{x}} = -\eta_0^* \tag{5}$$

where η_0^* and τ_0^* are constants. This paradigm problem may be solved in closed form. By integrating and using the obvious analogous "mass flow" condition to (3), we find that

$$\left(\frac{1}{\pi} \int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi - \bar{x}} d\xi \right)_{\bar{x}} - \tau_0^* \bar{x} = \eta_0^*(1 - \bar{x}). \tag{6}$$

Further rearranging, integrating, inverting using standard methods (see, for example [Mus53]), applying the boundary condition $\bar{h}'(0) = 0$ and finally integrating again and applying $\bar{h}(0) = 1$ and $\bar{h}(1) = 0$, we find that

$$\bar{h}(\bar{x}) = \frac{\sqrt{\bar{x}(1-\bar{x})}}{48} \left[-16K\bar{x}^2 + \bar{x}(-24\eta_0^* - 8K) + \frac{96}{\pi} \right] - \frac{\sin^{-1}(2\bar{x} - 1)}{\pi} + \frac{1}{2} \tag{7}$$

where $K = (\tau_0^* - \eta_0^*)/2$. It may easily be shown that (7) is the unique solution to (5) that satisfies the conditions (2) and the analogous condition to (3); we do not expect, of course that it will be possible to establish uniqueness results for the solution to (1). We may now use the extra condition (4) to determine L: we find that

$$L = \frac{h_0 \rho_\infty U_\infty^2}{16(p_\infty - p_{g0})} \left[\frac{32}{\pi} + 3\eta_0^* + \tau_0^* \right]. \tag{8}$$

Even though there is no reason why solutions to the paradigm problem should mimic solutions to the full problem in any way, it is evident that (8) mirrors much of the behaviour that we might expect from the solution to (1).

4 Numerical Solution of Paradigm Problem

We now discuss the numerical solution of (5). Many accurate and efficient methods exist for solving LSIDEs (see, for example [AdL80], [Gol78] and [Kre75]), but they rely specifically on the linearity of the equation. We shall therefore use a numerical method that generalises easily to the full nonlinear equation (1). Very little theory is available regarding the numerical solution of NLSIDEs, so we proceed in an *ad hoc* fashion and solve (6) using a finite difference method. We divide the interval $[0,1]$ into N equal subintervals $[\xi_j, \xi_{j+1}]$ where $0 \le j \le N-1$ and $\xi_j = j/N$ and write (6) as

$$p_{\bar{x}} = (\tau_0^* - \eta_0^*)\bar{x} + \eta_0^* \quad \text{where} \quad p = \frac{1}{\pi} \int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi - \bar{x}} d\xi$$

We then collocate at the internal half-mesh points and write

$$p_{\bar{x}}|_i = \frac{p_{i+1/2} - p_{i-1/2}}{\xi_{i+1/2} - \xi_{i-1/2}} = (\tau_0^* - \eta_0^*)\bar{x}_i + \eta_0^* \quad (1 \le i \le N-1) \tag{9}$$

where

$$p_{i+1/2} = \frac{1}{\pi} \int_0^1 \frac{\bar{h}_\xi(\xi)}{\xi - \bar{x}_{i+1/2}} d\xi$$

and $\bar{x}_{i+1/2} = (i+1/2)/N$. We now assume that in each subinterval $[\xi_j, \xi_{j+1}]$ the derivative of $\bar{h}$ is constant, so that

$$\bar{h}(\xi) \simeq \bar{h}_j + \left(\frac{\bar{h}_{j+1} - \bar{h}_j}{d\xi} \right) (\xi - \xi_j) \quad (\xi \in [\xi_j, \xi_{j+1}])$$

(where $d\xi = 1/N$) and introduce the fictitious mesh point $\bar{x}_{-1}$. A total of $N-1$ equations are provided for the $\bar{h}_i$ by (9) and three more come from the boundary conditions that $\bar{h}_0 = 1$, $\bar{h}_N = 0$ and $\bar{h}_{-1} = \bar{h}_1$. All integrals may now be calculated in closed form. The final collocation scheme becomes

$$\sum_{k=0}^{N-1} \left(\frac{\bar{h}_{k+1} - \bar{h}_k}{\pi d\xi^2} \right) \ln \left| \frac{(2k-2i+1)^2}{(2k-2i-1)(2k-2i+3)} \right| = (\tau_0^* - \eta_0^*) i d\xi + \eta_0^* \tag{10}$$

for $1 \le i \le N-1$ which, with the boundary conditions, gives an $(N+2) \times (N+2)$ system of equations for the $N+2$ unknown values of the $\bar{h}_i$. Unfortunately, though this scheme appears logical, it may easily be shown that in practice a saw tooth instability pervades for all N and the scheme is quite useless.

For many NLSIDEs the design of numerical methods must be guided by the asymptotic properties of the solution. For the paradigm problem (5) we note that the derivative of $\bar{h}$ becomes unbounded like $(1-\bar{x})^{-1/2}$ as $x \to 1$; it is this feature of the problem that renders the previous numerical method useless. Regularisation is therefore required (see, for example [KT93]). We redefine the variables so that the slope of $\bar{h}$ is finite at $\bar{x}=1$ and set

$$\bar{h}(\bar{x}) = \bar{H}(\bar{y}) \quad \text{where} \quad \bar{y}^2 = 1 - \bar{x}$$

giving the regularised paradigm equation

$$\left(\frac{1}{\pi}\int_0^1 \frac{\bar{H}_\xi(\xi)}{\xi^2-\bar{y}^2}\,d\xi\right)_{\bar{y}} = -2\bar{y}(1-\bar{y}^2)(\tau_0^* - \eta_0^*) - 2\bar{y}\eta_0^*$$

with boundary conditions $\bar{H}(0) = 0$, $\bar{H}(1) = 1$ and $\bar{H}_{\bar{y}}(1) = 0$. We may now design a simple numerical method. The analogous scheme to (10) is, for $i = 1, 2, \ldots, N-1$,

$$\frac{1}{\pi}\sum_{k=0}^{N-1}\left(\frac{\bar{H}_{k+1}-\bar{H}_k}{d\xi^3}\right)\left[\frac{1}{2i+1}\ln\left|\frac{(2k-2i+1)(2k+2i+1)}{(2k+2i+3)(2k-2i-1)}\right| - \right.$$

$$\left.\frac{1}{2i-1}\ln\left|\frac{(2k-2i+3)(2k+2i-1)}{(2k+2i+1)(2k-2i+1)}\right|\right] = -4Kid\xi(1-i^2d\xi^2) - 2id\xi\eta_0^*$$

with $\bar{H}_0 = 0$, $\bar{H}_N = 1$ and $\bar{H}_{N+1} = \bar{H}_{N-1}$. This scheme is simple to implement and gives very accurate results; for example, for the case $\eta_0^* = 1/2$, $\tau_0^* = 1$ the numerical results are almost identical to the exact solution even for as few as 50 points. We conclude that for NLSIDEs with singular asymptotic behaviour at the end points regularisation is essential.

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