MIXED HYPERBOLIC-ELLIPTIC SYSTEMS IN INDUSTRIAL PROBLEMS

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1.Introduction

In recent years there has been a large increase in the volume of literature pertaining to 'mixed' systems of partial differential equations In order to fix the terminology which we shall use, consider a system of conservation laws

$$w_t + Aw_x = S \tag{1}$$

where A is an $N \times N$ matrix, w is a N-vector, S is an N-vector of source terms and subscripts denote differentiation When the eigenvalues of A are non-zero, real and distinct, the system 1 is hyperbolic and many (though by no means all) of its mathematical properties are known (see, for example SMOLLER (1983)). In the strictest sense, the system 1 is said to be of 'mixed type' if any of the hyperbolicity conditions on the eigenvalues of A fail, so that for example two eigenvalues become equal, or an eigenvalue has a non-trivial zero. We shall be concerned mainly however with the much more serious case when some or all of the eigenvalues of A become complex, so that the problem is 'elliptic in time' In this case it is tempting to discard the problem completely, arguing that the innate ill-posedness of the system and need for boundary conditions at $t = \infty$ precludes any meaningful analytical or numerical results The fact remains however that such systems occur with surprising regularity. Because of this, a number of questions can be posed ·

(a) Can such complex sound speeds ever occur in a *correct* mathematical model ?

(b) Even if complex eigenvalues *are* present, do they occur in regions of phase space which the model ever enters ?

(c) If complex eigenvalues can never occur in 'correct' models, is there ever any point in studying mixed problems for 'physical' systems ?

(d) If we do study such problems and ultimately attempt to find a numerical solution, what effects can we expect to arise from the ellipticity ?

As far as (a) above is concerned, we can be fairly sure that equations which turn out to be mixed in some regions have a 'mistake' somewhere in them Although we distinguish between mixed systems where in some regions of phase space *all* of the eigenvalues are complex, so that

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the region is truly elliptic, and 'semi-elliptic' systems where only some of the eigenvalues become complex, leaving at least some information propagators, both must be considered to provide condemnation of our model What is more difficult however in many cases is to see how the model should be changed. Often in a mixed system the physical assumption which has led to ellipticity is a constitutive one In the equations of gas/particulate flow which we study in detail below, there is no doubt that the 'wrong' assumption is that a single pressure characterizes both phases - the pressure on the surface of a sphere moving through an inviscid fluid is simply not the same as the pressure far away from the sphere. Although modelling work is continuing with the aim of representing the interfacial pressure terms correctly, the fact is that at present the only equations available to us are 'incorrect' ones As far as (b) above is concerned, it is certainly true that in some mixed systems the elliptic regions are unlikely ever to be entered, though it should be remembered that numerical solutions might nevertheless stray into them due to discretization errors As we shall shortly see however for some equations the 'forbidden' regions cannot be avoided. It clearly makes sense therefore to study both physical mixed problems and also prototype systems of such equations. The points that we have made so far provide a rapid answer to (c); although we realize that a mixed system of conservation laws has serious faults, it may represent the best physical model which is available. To consider any worthwhile problems, we have to use the raw materials, however poor in quality they may be, until our modelling skill improves to a level where the equations can be trusted.

What effects do complex regions of phase space have on numerical schemes ? It is no surprise that the news here is nearly all bad A quick consideration of the system

$$w_t + A(x_0, t_0)w_x = 0$$

where the matrix A has been 'frozen' at some time t_0 and position $x_0$shows that if in the usual way we let P be the matrix of eigenvectors of A and D be the diagonal matrix composed of the eigenvalues of the 'frozen' matrix A, so that $A = PDP^{-1}$, then defining $\boldsymbol{w} = P^{-1}\boldsymbol{v}$ allows us to diagonalize the system Taking the complex Fourier transform shows that the differential equation for V_i, the transform of v_i is

$$V_{it} + \lambda_i ikV_i = 0$$

Under normal circumstances the solutions to this are purely oscillatory, but if one of the eigenvalues is complex then either it or its conjugate will inevitably lead to an exponentially growing mode. Thus any numerical inaccuracies will grow and eventually swamp the correct solution, their rate of growth being determined by the size of the imaginary part of the complex eigenvalue. Needless to say, it may also be shown that schemes which are TVD for hyperbolic systems become non-monotone. The only good piece of news is that, somewhat surprisingly, there are circumstances where strong source terms may help the stability when the eigenvalues are complex. This is in contrast to the situation generally encountered for strictly hyperbolic systems.

2. Mixed Systems in Industrial Problems

The discussion above has provided the motivation to study mixed problems, and this has been undertaken for both physical and prototype systems of conservation laws. One of the first systems to be recognized as mixed was the traffic flow problem described in BICK & NEWELL (1961) Whilst realizing the significance of the complex eigenvalues, they only considered the hyperbolic region, remarking that 'As yet we have found no satisfactory explanation of why the equations should be elliptic, nor any satisfactory suggestion as to what one should do about it'. JAMES (1980) considered a mixed system arising from waves in elastic bars, whilst HATTORI (1986) considered the flow of a Van Der Waal's fluid which led to a mixed system, showing that the concept of an 'admissible' solution could be physically motivated. Perhaps the most important contribution to the subject was that of BELL, TRANGENSTEIN & SCHUBIN (1986) who considered a model for the three phase flow of oil, gas and water in a porous medium. Their numerical calculations showed that when the initial states of a Riemann problem were chosen to be outside the elliptic region, the solution appeared to remain in the hyperbolic region for all time. They also found stable shocks connecting states inside the non-hyperbolic region with states outside.

As well as these physical problems, many authors have considered prototype mixed problems, most using the classical solution of the Riemann problem for 2×2 systems of conservation laws described in LIU (1974,1975) as a starting point. KEYFITZ & KRANZER (1983) studied Riemann problems for nonstrictly hyperbolic 2×2 systems where although the eigenvalues were real, they coalesced at an umbilic point. They found that in contrast to the strictly hyperbolic case where two waves are always sufficient, as many as four were necessary to construct a solution. A peculiar feature of some solutions was that for a given left state, the Hugoniot locus of the state was not necessarily connected to the state itself. This was also noted by SHEARER (1982) who solved a class of mixed conservation laws, producing a solution which involved stationary shocks not possessing viscous profiles, which left some doubts about their admissibility. SHEARER, SCHAEFFER, MARCHESIN & PAS-LEME (1986) identified 'undercompressive' shocks which did not fully satisfy the Lax entropy condition, but for which viscous profiles could be found More recently HOLDEN (1987) studied a mixed problem which was elliptic in a closed region of phase space She found that although the system always had a weak solution which could be constructed in the normal way, it was not unique Finally, mention should be made of the work of ISAACSON & TEMPLE (1985) and SCHAEFFER & SHEARER (1987) who have made a concerted attempt to solve and classify the general 2×2 system of non-strictly hyperbolic conservation laws with quadratic flux functions

Interest in mixed systems, both of elliptic and non-strictly hyperbolic type, is therefore great The theoretical work shows us that some novel numerical and analytical phenomena may be expected. To show that these are actually encountered in the study of physical systems of conservation laws, we now consider a specific example.

3. Two-Phase Gas/Particulate Flow

Let us now turn our attention to a specific mixed system, namely the equations of quasi one-dimensional two phase gas/particulate flow. These have been studied at length with the particular defence application of internal ballistics in mind In this scenario, a highly energetic solid burns, becoming part of a gas phase. The classical equations which have been used by many authors are

$$\begin{aligned}
(\rho_1 A_1)_t + (\rho_1 A_1 u_1)_x &= m_p + \dot{m} \\
(\rho_2 A_2)_t + (\rho_2 A_2 u_2)_x &= -\dot{m} \\
(\rho_1 A_1 u_1)_t + (\rho_1 A_1 u_1^2)_x &= -A_1 p_x - D + \dot{m} u_2 \\
(\rho_2 A_2 u_2)_t + (\rho_2 A_2 u_2^2)_x &= -A_2 p_x + D - \dot{m} u_2 - S_x \\
(\rho_1 A_1 E_1)_t + (\rho_1 A_1 u_1 \left(E_1 + \frac{p}{\rho_1}\right))_x &= -p(A_2 u_2)_x + m_p e_p - u_2 D + \dot{m} E_2 \\
&\quad - q - Q \\
(N_2)_t + (N_2 u_2)_x &= 0
\end{aligned}$$

Here the subscript 1 refers to the gas phase, 2 to a solid phase, which we take to be a granular material and p to the primer gas which initiates combustion We also use $S =$ intergranular stress, $D =$ interphase drag, $q =$ interphase heat transfer, $Q =$ heat loss to surrounding solid boundaries and $\dot{m} =$ gas mass addition due to burning, all of these terms being specified by various constitutive laws Also we have $E = \frac{u^2}{2} + e$ where e is the internal energy and $N_2 =$ number density of solid particles per unit length

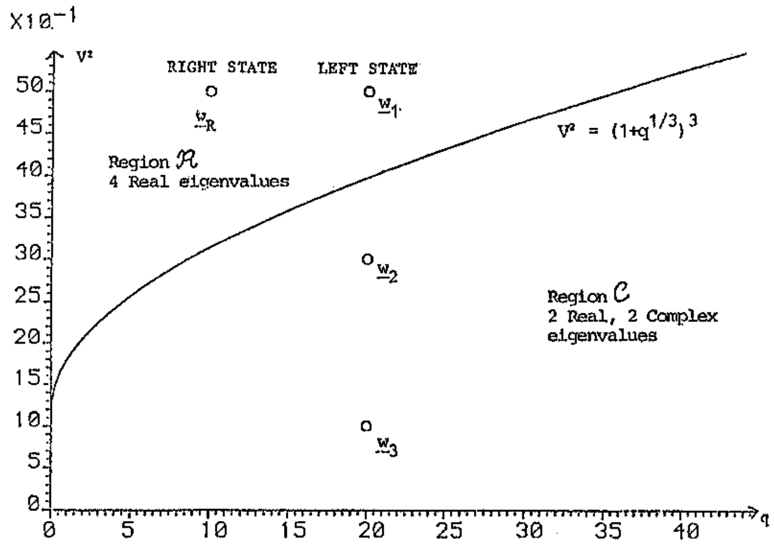

Figure 1 : Left and right states used in computations.

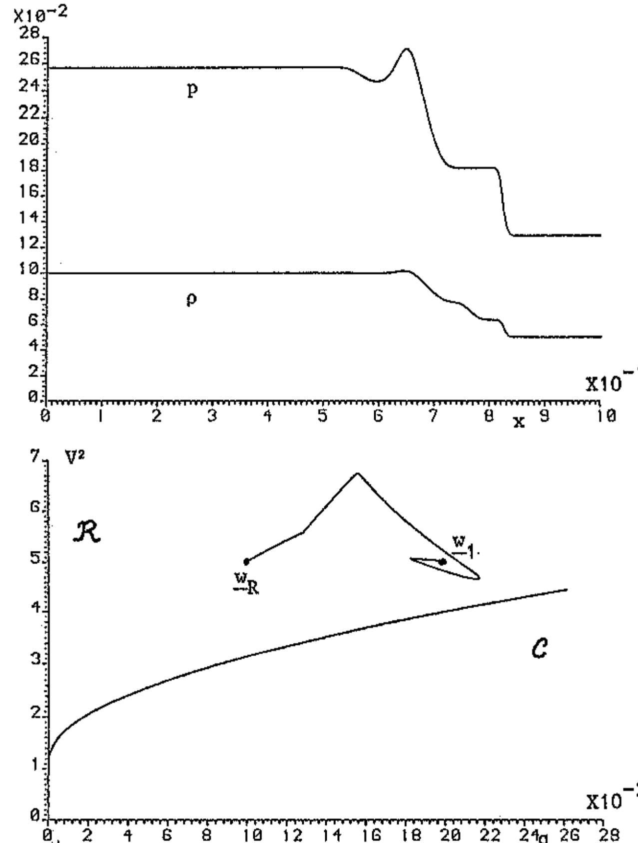

Figure 2 : profiles and phase path diagram for left state $\underline{w}_1$

To this model we must also add an equation of state, which we take to be the co-volume gas law relating pressure to temperature and density, and suitable boundary conditions.

For the forthcoming discussion, it is convenient to produce a simpler (though identical in type) prototype system of equations by setting all the source terms to zero, assuming an ideal gas and an incompressible solid and allowing motion to take place only within fixed boundaries (Normally one boundary will represent the base of an accelerating projectile so that the mesh expands). The equations become

$$\begin{aligned} (\rho_1 A_1)_t + (u_1\rho_1 A_1)_x &= 0 \\ (A_2)_t + (u_2 A_2)_x &= 0 \\ (u_1)_t + u_1 u_{1x} + p_x/\rho_1 &= 0 \\ (u_2)_t + (u_2^2/2 + p/\rho_2)_x &= 0 \\ (p)_t + u_1 p_x + (p\gamma/A_1)(u_1 A_1 + u_2 A_2)_x &= 0 \end{aligned} \tag{2}$$

It should be noted that the equations are not in conservation form This is a consequence of the averaging which has been used and is unavoidable Proceeding in the standard manner, we find that the eigenvalues of the system are given by $\lambda = yc + u_1$ where $c^2 = \gamma p/\rho$, and y satisfies the equation,

$$y(y^4 - 2Vy^3 + y^2(V^2 - q - 1) + 2Vy - V^2) = 0$$

where $V = (u_2 - u_1)/c$ and $q = (A_2\rho_1)/(A_1\rho_2)$ Clearly the zero sound speed corresponds to the incompressibility of the solid phase, but the behaviour of the roots of the remaining quartic is less clear For $V = 0$ (both phases moving with the same velocity) we find trivially that there are two more zero and two real roots, but some elementary analysis shows that for non-zero V the equation has four real roots if and only if $V^2 > (1 + q^{\frac{1}{3}})^3$, and if this condition is not met then there are two real and two complex sound speeds. A diagram of the region in (q, V^2) phase space is shown in figure (1), and we note that for any q as soon as there is a relative velocity we must pass through a semi-elliptic region $\mathcal{C}$ of phase space before reaching a strictly hyperbolic region $\mathcal{R}$ when the relative velocity is large enough

4. Numerical Results

To study the effects of the complex eigenvalues on numerical solutions to the prototype system, it is convenient to consider various Riemann problems where initially a 'left state' $w = w_L$ and a 'right state' $w = w_R$ are separated by a 'membrane' which bursts at time $t = 0$ Instead of using

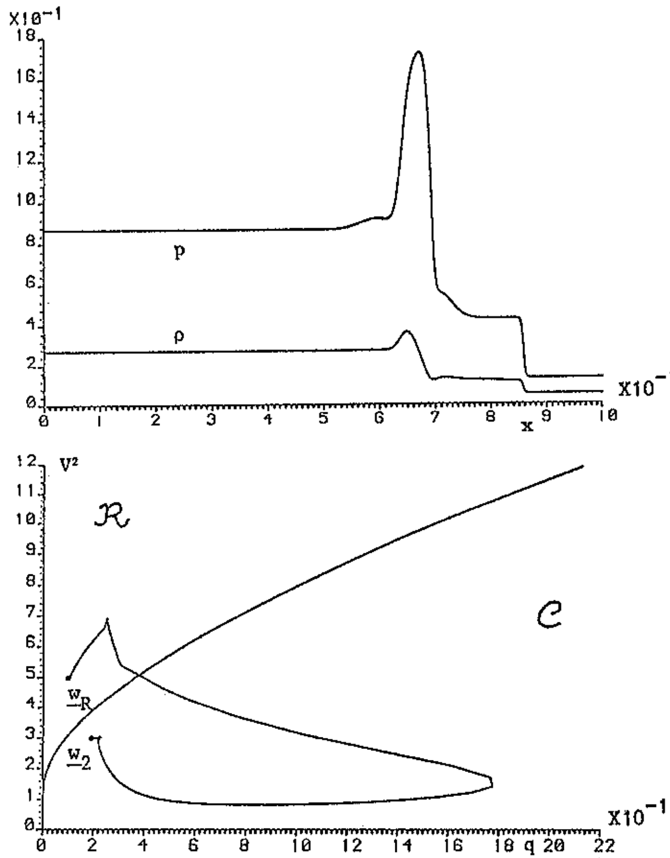

Figure 3 : profiles and phase paths for left state $\underline{w}_2$

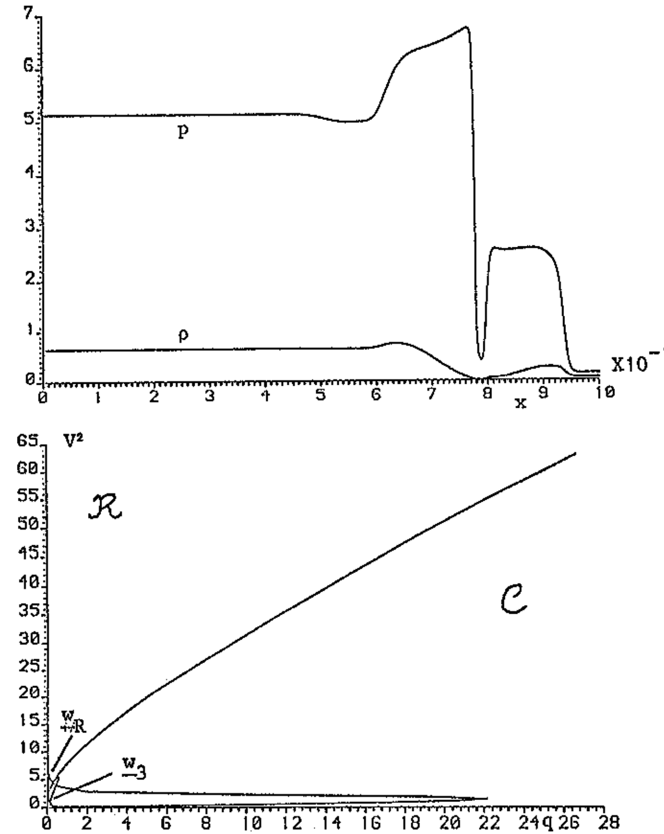

Figure 4 : profiles and phase paths for left state $\underline{w}_3$

TVD methods or other more sophisticated schemes, the very simple Lax - Friedrichs scheme was used so that phenomena arising from the existence of the complex eigenvalues could be clearly distinguished. The conclusions apply however to any scheme which may be used Of course the Lax-Friedrichs scheme is first order, and so for accuracy many (typically of the order of 1000) mesh points must be used The scheme is also monotone (that is, free from any spurious oscillations) for strictly hyperbolic problems, so that any non-monotonic effects which we see must either be part of the actual solution, or numerical consequences of the complex eigenvalues

Calculations were performed with 1000 space steps and a Courant number of 0.9, for a fixed right state

$$w_R = (\rho_1 A_1, A_2, u_1, u_2, p)^T = (0.05, 0.5, 5.0, 2.0, 0.12857)^T$$

and three different left states (see figure (1)) which started outside, moved a little way into $\mathcal{C}$, and finally lay well within $\mathcal{C}$

The results of these computations at time $t = 0.05$ are shown in figures (2) to (4) respectively In figure (2) where both initial states are in $\mathcal{R}$, it is clear that a smooth, physical solution is obtained, which is, as one would expect, similar in many ways to a solution of the standard shock tube gas dynamics problem In figure (3) the left state lies just inside $\mathcal{C}$, and whilst the solution still looks reasonably physical, the phase diagram shows that the complex region has been entered, and some puzzling non - monotonicities are present in the pressure and density profiles These occur near to $x = 0.7$ and are magnified in figure (4) as the left state penetrates deeper into $\mathcal{C}$. The velocity and phase diagrams for these two final cases are also most revealing - a potentially disastrous amplification is taking place and the phase path is extending further and further away from the left and right states into the complex region. It need hardly be said that the solution of figure (4) looks most unphysical.

To display the problems caused by the complex eigenvalue region even more clearly, figure (5) shows the phase diagram for a calculation made with both states very near to the right state $w_R \in \mathcal{R}$ used above As expected, the problem exhibits 'well-posedness' in that the phase path remains close to both states The situation in figure (6), when the states are close to each other but lying near to $w_3 \in \mathcal{C}$ is very different As time progresses, the pressure develops a bizarre shock system, which later computations (not shown) reveal to be a stable propagating profile (though in some circumstances growing unboundedly) The phase diagram of figure (6) confirms this - the phase paths wander, via a sequence of shocks and rarefactions, ever further away from the originally adjacent left and right states It is worth pointing out the contrast between these

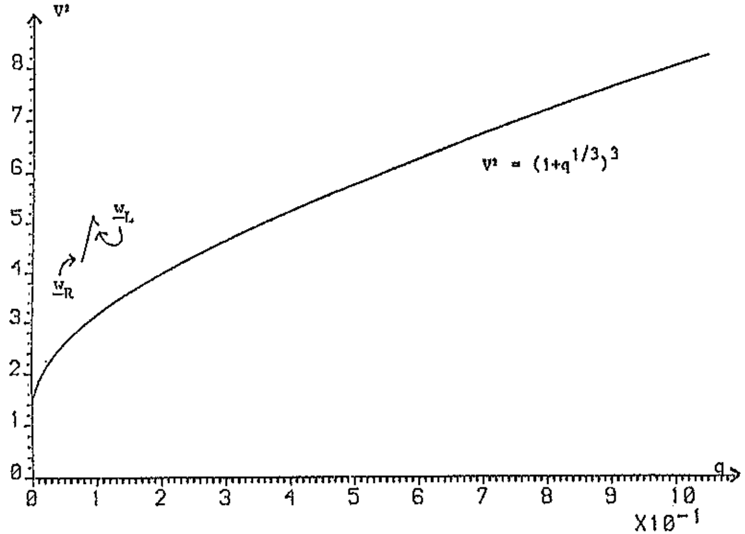

Figure 5 : Phase diagram for case where left and right states are taken close to each other in the strictly hyperbolic region. Solution clearly stays close to both states.

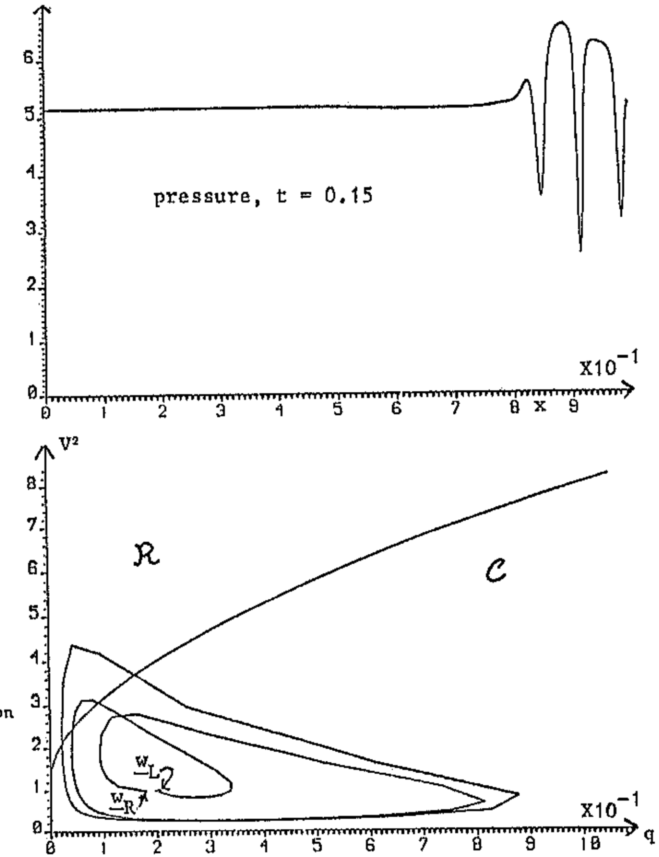

Figure 6 : pressure profile and phase diagram for case where states are adjacent in complex region

results and those of Bell, Trangenstein & Schubin, whose region of ellipticity was closed

4. Conclusions and Discussion

We have seen that the equations of quasi one-dimensional two-phase flow which are frequently used as the basis for numerical calculations of internal ballistics flows have complex eigenvalues under certain conditions which are physically realisable Experience has shown that numerical problems in internal ballistics codes have often been accompanied by profiles similar to those in figure (6) Space does not permit a full discussion of all the mixed problems which we have studied, but the main point of this paper is to encourage the study of mixed systems in their own right. Certainly models which lead to mixed systems have their faults, and in some cases the very existence of the complex eigenvalues may point the way to better modelling For complicated problems however, often we have to make the best of the equations which we have An alternative to this is is to add equations piecemeal to the system until we arrive at a totally hyperbolic system This has been attempted by many authors and usually results in a model which is unphysical in some regions of phase space. Surely it is preferable to use equations whose underlying assumptions and derivation are clear, even if they do contain some complex eigenvalues All that we must ensure is that their limitations, faults and possible associated problems are well understood. This last caveat, after all, is the litany on which all mathematical models are based.

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