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INJECTION MOULDING OF KEY BLANKS WITH ZINALCO

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1 Introduction

Zinalco is an alloy of Zinc (77%), Aluminium (21%) and Copper (2%) that has many desirable properties as far as manufacturing is concerned. Products made with zinalco have excellent strength, and the alloy is particularly easy to work with as it has a low melting temperature. Professor Torres (Instituto de Investigaciones en Materiales, UNAM) has developed zinalco extensively, and is now considering manufacturing processes that use the alloy This study specifically concerns the manufacture of household key blanks by injection moulding. In a typical production process, a triangular distribution chamber is connected to five key-shaped chambers. The distribution chamber is filled, and molten zinalco is forced into the key moulds by a pressure gradient. (See figure (1) for a schematic diagram.) The total time of injection (including the filling of the distribution chamber) is typically 10 seconds. A number of different types of key may be made in this way, but the dimensions of a typical key may be thought of as being about 5cm×5cm×2mm, there being little variation in the vertical dimension Ideally, the injection moulding process should be capable of a 'quasi continuous' mode of operation where, when the mould is opened soon after injection is completed, a solid object is ready for further processing.

The melting temperature of Zinalco may be taken to be about 421-481 C. The variation is due to the fact that the alloy is a two-phase material, so that it is difficult to identify a unique phase-change temperature. In any case, the alloy is completely liquid at 481 C. In order that the process may be quasi continuous as described above, the walls of the mould are water-cooled. Although the amount of water that is used may be varied for different injection rates, it may be assumed that the mould walls are maintained at a temperature of between 150 and 200 C. In its liquid state, the density, thermal conductivity and specific heat of zinalco may be assumed to take the values $\rho = 5000$ kg/m^3, $k = 85$W/m/C and $c_p = 0\ 56$ kJ/kg/C respectively.

The main questions that have to be answered before the process can be considered to be completely reliable include:

- The air that is displaced by the injection moulding process escapes from the mould via strategically placed 'chimneys' Where should such chimneys be placed in order to ensure that all air is efficiently ejected from the mould ?

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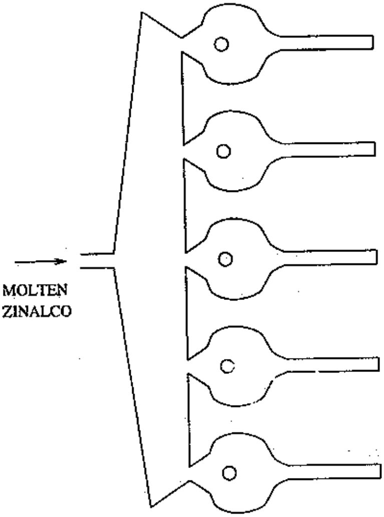

Figure 1: Schematic diagram of key blank moulding

- Can air bubbles arising from the passage of the injection front around obstacles in the flow (for example, holes moulded in the key to allow the key to be attached to a ring) be avoided in the final product ?
- If the mould walls are cooled, will the alloy freeze before the injection moulding process is completed ?
- What parameter values are likely to lead to 'efficient' freezing where the key is solid a short time after the injection has been completed ?
- For complicated geometries, are there simple numerical techniques that will allow the mould filling process to be calculated and optimised ?

It is worth pointing out that it is key *blanks* that are to be made. The geometry is therefore relatively simple in most applications and there are few tortuous parts of the mould into which the molten zinalco must be injected. Of course from a security point of view it would be pointless to produce large numbers of completely identical keys; When the key blanks are solid, therefore, they are sent for further processing where they may be machined into their final shape.

2 Isothermal Injection Moulding

We begin by considering the simplest injection moulding problem where isother-

isothermal problem may be posed in such a manner to allow further complications to be introduced later. We assume that the mould occupies the region $0 \leq x \leq L, 0 \leq y \leq L, -h \leq z \leq h$ and that the dimensions of the mould are such that $\epsilon = h/L \ll 1$ is a small parameter. Beginning with the Navier-Stokes equations

$$q_t + (q.\nabla)q = -\frac{1}{\rho}\nabla p + \nu\nabla^2 q$$

$$\nabla q = 0$$

where $q = (u, v, w)$ denotes the fluid velocity, p the pressure and ν the kinematic viscosity of molten zinalco (assumed constant) we introduce the standard 'Hele-Shaw cell' non-dimensionalisation (see, for example Ockendon & Ockendon [1]). Setting $x = L\hat{x}$, $y = L\hat{y}$, $z = \epsilon L\hat{z}$, $t = (L/U)\hat{t}$, $u = U\hat{u}$, $v = U\hat{v}$, $w = \epsilon U\hat{w}$ and $p = (\mu U/L\epsilon^2)\hat{p}$ where μ denotes dynamic viscosity and U is a typical fluid velocity, and dropping the hats immediately for convenience, we obtain

$$\epsilon^2 Re[u_t + uu_x + vu_y + wu_z] = -p_x + \epsilon^2(u_{xx} + u_{yy}) + u_{zz}$$

$$\epsilon^2 Re[v_t + uv_x + vv_y + wv_z] = -p_y + \epsilon^2(v_{xx} + v_{yy}) + v_{zz}$$

$$\epsilon^4 Re[w_t + uw_x + vw_y + ww_z] = -p_z + \epsilon^2(\epsilon^2(w_{xx} + w_{yy}) + w_{zz})$$

$$u_x + v_y + w_z = 0$$

where $Re = UL/\nu$ is the usual Reynolds number. To leading order, therefore, the equations of motion are

$$p_x = u_{zz}, \qquad p_y = v_{zz}, \qquad p_z = 0, \qquad u_x + v_y + w_z = 0,$$

this simplification being valid so long as $\epsilon \ll 1$ and $\epsilon^2 Re \ll 1$. With the values given above $\epsilon = h/L \sim 0.04$ and, assuming that $U \sim$ 5cm per 10 sec $= 5 \times 10^{-3}$m/sec and $\nu \sim 10^{-5}\text{m}^2/\text{sec}$, we find that $\epsilon^2 Re \sim 4 \times 10^{-2}$ so that the Hele-Shaw approximation is a valid one. (The value for viscosity is a guess based on a value of $\mu = 5 \times 10^{-2}$ kg/m/sec, but it seems most unlikely that molten zinalco will be less viscous than water which at room temperature has $\mu \sim 10^{-3}$ kg/m/sec. When zinalco was being considered for use in continuous casting an experiment was carried out where the time taken for molten zinalco at 500 C to flow out of a 40mm by 75 mm crucible through a 1mm hole was measured. The results indicated that the viscosity was comparable to that of water.)

Solving the first two equations for u and v now gives (on application of the no-slip boundary condition at $z = \pm 1$)

$$u = -\frac{(1 - z^2)p_x}{2}, \qquad v = -\frac{(1 - z^2)p_y}{2}$$

and an equation for p may speedily be determined by integrating the mass conservation equation between $z = -1$ and $z = 1$. The result is that the pressure must satisfy the two-dimensional Laplace equation

$$p_{xx} + p_{yy} = 0$$

The variables u and v may also be integrated across the cell, giving average velocities

$$\bar{u} = -\frac{1}{3}p_x, \qquad \bar{v} = -\frac{1}{3}p_y$$

and it is usual at this stage to make the observation that the pressure resembles the velocity potential commonly encountered in inviscid incompressible irrotational two-dimensional flow since it satisfies Laplace's equation and and the (average) velocities are proportional its gradient.

The whole problem may now be posed in terms of the pressure At the mould boundaries the normal derivative of the pressure must be zero, whilst at the injection boundary the pressure is constant ($p = 1$, say). At the free boundary where the injection front is situated, the pressure must also be constant ($p = 0$, say) and the extra condition that allows the position of the front to be determined is simply that the normal derivative of the pressure is proportional to the front speed This formulation of the problem is particularly convenient to work with numerically since (i) a three-dimensional problem has been replaced by a two-dimensional one and (ii) time does not appear explicitly in the equation, so that the problem is quasi-steady. The normal strategy is to solve Laplace's equation at a given time (many methods may be used to do this, but the boundary element method is particularly efficient and convenient), use the solution thus computed to to advance the free boundary, and solve Laplace's equation again on the new solution region. For extra convenience the condition on the free boundary may be expressed solely in terms of the pressure by noting that setting $p = 0$ on the free boundary means that the boundary itself is determined by $p(\boldsymbol{x}, t) = 0$. If this is still to be the free boundary after a short time interval dt, then

$$p(\boldsymbol{x} + v_n \boldsymbol{n} dt, t + dt) = v_n dt \boldsymbol{n}.\nabla p + p_t dt = 0$$

where v_n is the normal velocity. Thus

$$v_n = -\frac{p_t}{\mid \nabla p \mid}$$

Since the normal velocity is also given by $v_n = -(\nabla p\, \boldsymbol{n})/3$, and $\boldsymbol{n} = \nabla p / \mid \nabla p \mid$, we find that on $p = 0$ we have

$$p_t - \frac{\mid \nabla p \mid^2}{3} = 0$$

In very simple cases, explicit solutions are available. For example, if injection takes place into a straight sided channel so that everything is independent of y and the injection front is given by $x = f(t)$, it is easy to confirm that the pressure is given by

$$p = 1 - x\sqrt{\frac{3}{2t}}$$

whilst

$$f(t) = \sqrt{\frac{2t}{3}}.$$

Thus in dimensional form

$$p = \frac{\mu U}{h^2}\left(L - x\sqrt{\frac{3L}{2Ut}}\right)$$

and the injection front advances according to

$$x = \sqrt{\frac{2UtL}{3}}$$

This result is as might be expected; the injection front slows as it advances and an increase in pressure at $x = 0$ would be required if it were to move a constant speed.

A great deal of work has been carried out on other aspects of isothermal injection moulding. Richardson [2] considered a range of problems where a blob of viscous fluid occupied a simply connected domain D_0 in the narrow gap between two plates. Additional fluid was assumed to be injected at some fixed point within D_0 so that the blob grew in size. His main motivation was to determine the regions that were filled last, so that the placement of air vents in the mould could be optimised. Using complex variable methods, Richardson was able to reduce the problem for simple initial domains to a finite system of algebraic equations. In a later study [3] Richardson was able to analyse flow in finite regions bounded by walls by using images. In this way some quite involved and realistic exact solutions could be obtained. For example, in the case of injection in a half-plane a blob of viscous fluid expands radially until it touches a wall. Richardson's methods may then be used to show that, as the blob expands along the wall, it takes up the shape of a Lemniscate of Booth [4]. Injection into a quarter-plane may also be examined in this way.

For injection moulding, our interest is solely in the problem where a pressure gradient forces viscous fluid *into* the mould. Perhaps the greatest volume of literature on the subject has concerned itself with the ill-posed problem where liquid is sucked *out* of a region. Instabilities that manifest themselves in the form of 'fingering' and cusp development are all well known and understood (see, for example Howison et al. [5]). Although some exact solutions are available here also, matters are greatly complicated by the topological changes in the evolving flow.

As well as the literature described above, it is worth pointing out that the usual assumptions that are made during discussion of the two-dimensional electromachining problem lead (see, for example Collett et al [6] and Fitz-Gerald & McGeough [7]) to equations and boundary conditions that are almost identical to free boundary Hele-Shaw cell problems

3 Non-isothermal Injection Moulding

Isothermal injection moulding is easy to analyse and involves none of the problems that are associated with premature freezing of the alloy which might prevent complete filling of the mould Unfortunately isothermal injection moulding also takes considerably more time as the molten zinalco must cool before the key blank can be removed from the mould. In order to achieve a more or less continuous process, where the key blank may be removed from the mould almost immediately after it has been filled, the mould walls may be cooled. In this case an energy equation must be added to the equations of motion and account must be taken of the fact that the viscosity varies with temperature Assuming that the thermal properties of the zinalco are constant and ignoring viscous dissipation, the new governing equations become

$$\boldsymbol{q}_t + (\boldsymbol{q}.\nabla)\boldsymbol{q} = -\frac{1}{\rho}\nabla p + \frac{\partial}{\partial x_j}\left(\nu\frac{\partial q_i}{\partial x_j}\right)$$

$$\nabla \boldsymbol{q} = 0$$

$$\rho c_p(T_t + (\boldsymbol{q}.\nabla)T) = k\nabla^2 T$$

Non dimensionalising in a similar way to the isothermal case by setting $x = L\hat{x}$, $y = L\hat{y}$, $z = \epsilon L\hat{z}$, $t = (L/U)\hat{t}$, $u = U\hat{u}$, $v = U\hat{v}$, $w = \epsilon U\hat{w}$ $p = (\mu U/L\epsilon^2)\hat{p}$, $\mu = \mu_0\hat{\mu}$ and $T = T_0 + \hat{T}(T_i - T_0)$ where T_0 denotes the mould wall temperature and T_i the temperature at injection from the triangular feeder, we find that to leading order (dropping the hats)

$$p_x = (\mu u_z)_z$$

$$p_y = (\mu v_z)_z$$

$$p_z = 0$$

$$u_x + v_y + w_z = 0$$

$$T_t + uT_x + vT_y + wT_z = \alpha T_{zz}$$

where

$$\alpha = \frac{k}{\ldots}$$

and the conditions for the approximations to be valid are once again that ϵ and $\epsilon^2\mathrm{Re}$ are both much less than one. The importance of the parameter α should be stressed as it measures the relative importance of diffusion and convection. For $\alpha \gg 1$ we have to leading order

$$T_{zz} = 0$$

and the temperature is therefore given by $T = T_0$ except in a region of width order h near to the original injection point. Convection is unimportant and, save for a small 'entry region' where the flow adjusts from T_i to T_0 the temperature (and thus the viscosity) is constant. Obviously if T_0 is near to the freezing temperature of the alloy this is a highly undesirable state of affairs as the zinalco will be able to penetrate into only a small part of the mould before freezing solid

Conversely, if the parameter α is small convection is all-important and diffusion plays only a small role The influence of the cooled mould is felt only in a small layer near to the boundaries and most of the flow remains at a temperature close to T_i. This essentially reproduces the isothermal case and probably cannot give a continuous process unless a fairly strong 'skin' was formed at the mould walls that allowed the key, though molten on its inside, to retain its shape after ejection from the mould

Having discussed the two obvious special cases of large and small α, we now note that in the case currently being considered α takes a value (based on the parameters given above) of about 76. This is somewhat surprising as it seems to indicate that the zinalco may freeze before it reaches the end of the mould. Further estimates of α should probably be made when more of the process details have been confirmed.

In general, the temperature dependent problem requires a numerical solution. There are various simplifications that occur in particular special cases, however. For example, if we neglect variations in the y-direction and assume that, at entry, the flow is steady and of fully-developed Poiseuille type, then on introducing a non dimensional stream function $\psi(x, z)$ with $u = \psi_z$, $w = -\psi_x$, the problem may be written

$$zp_x = (\mu\psi_{zz})$$

$$\psi_z T_x - \psi_x T_z = \alpha T_{zz}$$

with boundary conditions

$$T = \psi_x = \psi_z = 0 \quad \text{at} \quad z = \pm 1,$$

$$T = 1, \quad \psi = z - z^3/3 \quad \text{at} \quad x = 0.$$

Considering first the entry region, we note that for x small (but larger than z) we may expect the velocity profile to be largely unchanged. The boundary layer at the top of the channel may now be analysed setting and seeking a local solution

$$\psi = \frac{2}{3} + x^k f(\zeta), \qquad T = g(\zeta)$$

we find that a similarity solution exists if $k = 2/3$ and $n = 1/3$ in which case we have

$$\zeta = \frac{z - 1}{(x\alpha^3)^{1/3}}$$

$$\mu(g) f'' = \alpha^2 p'(0), \qquad 3g'' + 2fg' = 0$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad g(0) = 0, \quad g(\infty) = 1.$$

A single equation for f may be obtained by solving for g to yield

$$g = \frac{\int_0^\zeta \exp(-\int_0^q \frac{2}{3} f(s) ds) dq}{\int_0^\infty \exp(-\int_0^q \frac{2}{3} f(s) ds) dq}$$

and further progress may be made for particular choices of viscosity; the main conclusion however is that, in the entry region where the pressure gradient is more or less constant, the wall thermal boundary layer grows like $x^{1/3}$. (The behaviour of the bottom boundary layer is, of course, identical.)

A number of other similarity solutions and different parameter regimes may also be examined, and high activation energy asymptotics are also possible for viscosities of certain types. For more details the reader is referred to Ockendon and Ockendon [8] where a similar problem was discussed.

4 Purely Numerical Studies of Injection Moulding

Although the simple models described above should be capable of predicting the main details of the moulding process, it is likely that, if the process is to be carefully optimised, detailed numerical calculations will be required. There is a considerable literature on injection moulding problems, and much of it concerns numerical attacks on the problem. For example, Ladiende and Akay [9] considered mould filling by injection of thermoset polymers, using the equations

$$\nabla\ (S\nabla p) = 0$$

$$\rho c_p(T_t + (q.\nabla)T) = kT_{zz} + \alpha_t Q_m + \eta\dot{\gamma}^2$$

$$\alpha_t + (q.\nabla)\alpha = \frac{d\alpha}{\quad}$$

where Q_m denotes the heat generation from the exothermic cure reaction, $\dot{\gamma}$ is the strain rate and α denotes the 'degree of cure' that has taken place in the polymer. The 'fluidity' S is related to the viscosity of the polymer and the term $d\alpha/dt$ on the right hand side of the final equation must be specified through a separate 'curing reaction' model. A number of computational issues are addressed in this study, but the basic conclusion is that the finite element method may be used to solve the problem to a satisfactory degree of accuracy.

In Subbiah et al. [10] the Hele-Shaw approximations are used, much as described above, to simplify the general flow equations. Numerical grid generation is then used to map the flow region onto a more regular computational domain. The equations may then be solved in a straight forward manner using standard finite difference techniques. It is shown that even highly irregular domain shapes may be handled in this way, and good agreement with experiment is obtained. This paper also contains numerous references to which the reader is referred for further numerical details.

5 Bubble Creation in the Moulding Process

Some discussion is worthwhile concerning the problem of bubble creation during the injection moulding process. A product with bubbles of any appreciable size is quite useless, as both the visual standard and the strength of the key are compromised. In normal use keys are often subjected to large turning moments and stresses (especially in old and ill-fitting locks). As anybody who has had the experience of seeing a key break off in a lock will appreciate, reductions in the strength of the final product cannot be tolerated.

Under normal circumstances, (for example, when the shape that is to be manufactured is simply connected) it may be expected that the last areas of the mould that are filled will be most prone to bubble development, and evidently 'chimneys' should be placed here. Detailed calculations using either the isothermal theory of section 2 or the more complicated temperature-dependent model of section 3 may be used to determine which areas of the mould are filled last, but for most key blanks it is clear that the region at the end of the key (furthest from the injection point) will almost certainly be the correct area to site 'chimneys'.

When keys are to be made with holes in them (as is normally the case), matters become altogether more complicated. The continuous injection boundary must split into two distinct pieces when a hole is encountered, and a danger arises that a bubble may be trapped behind the obstacle as the distinct injection boundaries remerge. Figure (2) shows two possible behaviours for the injection boundary. Obviously in the first case (i) a bubble will be trapped at the rear of the circular hole, whilst in (ii) no bubble will be formed. Although in most simple models of the injection moulding process the bubble will eventually close of its own accord, when conservation of mass inside the bubble is included a

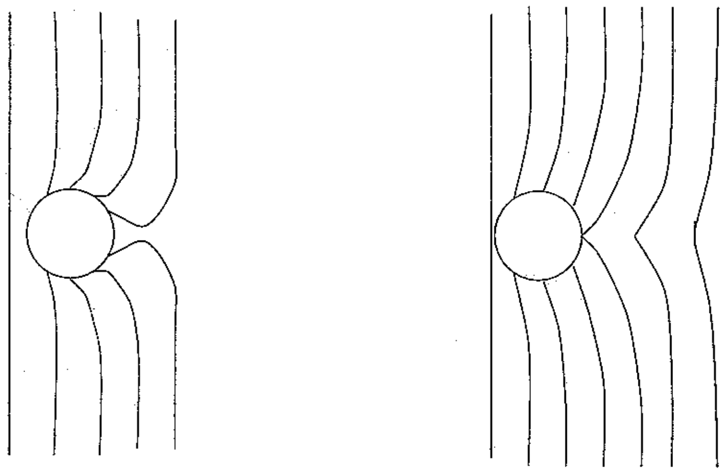

Figure 2: Bubble creation and bubble avoidance free boundary topologies

permanent bubble will be formed when the internal pressure becomes too high. Primarily because of the changes in topology, it seems to be very hard to analyse such bubble trapping; multiply connected shapes are also likely to complicate any numerical methods that are being used. The only sensible precaution seems to be to place chimneys at the rear of obstacles so that bubbles that may form may be removed.

6 Conclusions

The following conclusions may be drawn concerning the injection moulding of zinalco:

- For isothermal injection moulding a simple mathematical model may be posed that would allow the mould filling to be determined completely using elementary methods
- Even in the non-isothermal case, the problem is considerably simplified by using asymptotic analysis; in particular only the temperature equation contains a time derivative and a quasi-steady approach may thus be taken.
- Mathematical descriptions of most flow regimes existing in water-cooled moulds may be developed using asymptotic methods
- In the non-isothermal case the parameters indicate that the possibility of solidification and subsequent mould blockage cannot be discounted

- The full non-isothermal problem may be analysed using fairly standard finite difference or finite element methods; a large literature already exists.
- The matter of bubble formation behind obstacles in the flow is a hard one to analyse, but it seems that placing chimneys at these positions would be a good idea.

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