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Crack Propagation in a Geothermal Energy Reservoir

We consider the problem of the opening of pre-existing cracks with high pressure fluid in the case where changes in the fluid pressure are balanced predominantly by stresses due to the local crack asperity deformations. A model is proposed for this process using a particular choice of 'crack law' and 'flow law' which leads to a free boundary problem. A singular perturbation analysis reveals that, in the boundary layer, the motion of the free boundary is governed by a singular integro-differential equation.

1. Introduction

Hot dry rock geothermal energy reservoirs (HDRGERS) generally consist of large networks of interconnected subterranean cracks, typically at a depth of a few kilometres. Cold water that has been pumped into the network from a borehole is heated by the hot rock and the resulting hot water is extracted at another borehole. To enhance mass flow, pre-existing cracks are further opened by pumping a viscous gel into the rock. To model the opening of such a crack, we assume that a one-dimensional analysis is appropriate (the aspect ratio of such cracks is invariably large) and further assume that any influence of heat transfer processes may be ignored as timescale for such changes is typically measured in years.

2. A mathematical model for crack propagation in a HDRGER

Using Cartesian coordinates with the direction x assumed to lie along the crack and y normal to it and denoting the crack width by $h(x,t)$, we assume that the rock is a linearly elastic medium and the fluid is incompressible and Newtonian with dynamic viscosity μ. The fluid pressure in the crack is denoted by $p(x,t)$, the normal stress in the rock by σ_{yy} and the reference stress (the least negative value of $p+\sigma_{yy}$ when $h=0$) by σ_R. For $h > h_{\max}$ a crack is assumed to be fully opened. It has also been assumed that the minimum crack height is zero; in practice this may not be true, since even at very large compressive normal stresses there will always be a residual aperture. However, it is invariably the case that $h_{\min}/h_{\max} \ll 1$, and this fact is used to justify the neglect of $h_{\min}$. Needless to say, this assumption inevitably gives rise to cracks with compact support. It may easily be shown, however (see, for example [4]) that if $h_{\min}$ is non-zero, the effect upon the solution is only exponentially small.

To model the crack, we adopt a piecewise linear crack law that was first suggested in [6]. According to this law, $h=0$ when $\sigma_{yy}+p \le \sigma_R$, whilst for $\sigma_R < \sigma_{yy}+p < 0$ we have

$$h = h_{\max} - (\sigma_{yy}+p)\frac{h_{\max}}{\sigma_R} \tag{1}$$

In effect, this law reflects the fact that in a typical HDRGER the fluid pressure alone is not enough to support the normal stress. Instead, the load is distributed between the fluid pressure and local deformations of asperities in the cracks. Other models are possible; for example, the 'hyperbolic' crack law first suggested in [2] has been frequently used. Analysis is possible for this case also, though space does not permit a discussion here. When this law is used, however, there are important differences in the mathematical techniques required to analyse the model. For full details the reader is referred to [1].

To determine the fluid flow in the crack, a modification of Reynolds' equation is used. Based upon experimental evidence, it seems that a flow law of the form

$$h_t = \frac{1}{12\mu}(a_n h^n p_x)_x \tag{2}$$

is the most accurate, where a_n and n are constant. For parallel plates, $a_n = 1$ and $n = 3$, however, for the type of crack that we wish to consider, experiments indicate that a value of n close to unity is appropriate, and accordingly we take $n = 1$. Once again, analysis is possible for other cases, some of which are considered in [1].

To close the model, we use the standard plane strain elastic contact problem (see, for example [5]) to write

$$\sigma_{yy}(x,t) - \sigma_{yy}^{\infty} = \frac{G}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial h(s,t)}{\partial s} \frac{ds}{s-x} \tag{3}$$

where σ_{yy}^{∞} is the normal stress at infinity in the rock and G and ν are respectively the rock shear modulus and Poisson ratio.

Using (1), (2), and (3), the quantities σ_{yy} and p may be eliminated. It is also convenient to non-dimensionalize the model by setting $h = h_{\max}\bar{h}$, $x = L\bar{x}$, $p = p_0 - (\sigma_{yy}^{\infty} + p_0)\bar{p}$, $\sigma_{yy} = \sigma_{yy}^{\infty} + (Gh_{\max}/2\pi L(1-\nu))$ and $t = -12\mu L^2/(\omega h_{\max}^2)\bar{t}$ where L is the crack length, p_0 is a typical fluid pressure and $\omega = a_1\sigma_R/h_{\max}^2$. Dropping the bars, the final equation to be solved is

$$h_t = \left[hh_x - \epsilon h \frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} \frac{\partial h}{\partial s}\frac{ds}{s-x}\right)\right]_x \qquad \text{where } \epsilon = \frac{-Gh_{\max}}{2\pi\sigma_R(1-\nu)L} \tag{4}$$

Denoting the non-dimensional crack width by $\ell(t)$, we consider the problem of the spreading of a fixed amount of fluid, in which case (4) must be solved subject to the conditions

$$h(x,t) = \begin{cases} 0 & (\ell(t) \le |x|) \\ > 0 & (-\ell(t) < x < \ell(t)) \end{cases}, \qquad h_x(0,t) = 0, \qquad \int_{-\ell(t)}^{\ell(t)} h(x,t)dx = 1.$$

These conditions assume that we seek to determine the large-time behaviour of the solution: the source condition may be thought of as a matching condition. We also insist that, since the crack is pre-existing, and thus does not possess a stress singularity at its tips, $h(x,t) = o(|\ \ell(t) - x\ |^{1/2})$ as $x \to \pm\ell(t)$. The non-dimensional parameter ϵ measures the relative importance of alterations in the deformation of the asperities and changes in the elastic normal stress in the rock and may take various values in different circumstances; for the typical applications that motivated the present study, however, it is of order 1/10 to 1/100, and hence the small ϵ case will be considered below.

3. Analysis of the mathematical model

Seeking a regular perturbation expansion of the form

$$h = h_0 + \epsilon h_1 + \ldots, \qquad \ell = \ell_0 + \epsilon\ell_1 + \ldots$$

in the normal way, we find that the leading order partial differential equation is

$$h_{0t} = (h_0 h_{0x})_x,$$

which has solution, (given for $0 \le \eta < 1$) by

$$h_0 = \frac{\lambda^2}{6t^{1/3}}(1-\eta^2), \qquad \ell_0(t) = \lambda t^{1/3} \quad \text{where } \lambda = \left(\frac{9}{2}\right)^{\frac{1}{3}}, \quad \eta = \frac{x}{\lambda t^{1/3}}.$$

It is instructive to consider the asymptotic behaviour as $\eta \to 1$ of the terms on the right-hand-side of (4). We find that

$$h_0 h_{0x} \sim 1-\eta, \qquad h_0\frac{\partial}{\partial x}\left(\int_{-\infty}^{\infty} \frac{\partial h_0}{\partial s}\frac{ds}{s-x}\right) \sim 1, \quad \text{as} \quad \eta \to 1$$

suggesting that a singular perturbation/matched asymptotic expansion analysis is required.

Away from the crack tips, the outer solution may be obtained by setting $h_1(x,t) = t^{-2/3}H(\eta)$ and observing that H satisfies the equation

$$(1-\eta^2)H'' - 2\eta H' + 2H = \frac{4\lambda}{3} - \frac{2\lambda\eta}{3}\log\left(\frac{1+\eta}{1-\eta}\right).$$

This has solution

$$H(\eta) = A\left(2 + \eta\log\left(\frac{1-\eta}{1+\eta}\right)\right) + h_p(\eta) \tag{5}$$

where h_p, though easily determined in terms of polynomials, logarithms and dilogarithms, is somewhat involved. A is an arbitrary constant, the other component of the complementary function having been discarded since it is odd.

Near the crack tips, where $\eta \sim 1-\epsilon$, (5) is of order $\log\epsilon$ and is therefore not valid. To determine an inner solution, we write $h_{\text{inner}} = h_0 + \epsilon\bar{h}_1 + \ldots$. This slightly unconventional *ansatz* is convenient in this case as it ensures that $\bar{h}_1$ remains bounded when matching takes place. Examining the solution near the crack tip $\eta = 1$, we set $x = \ell - \epsilon X$, $t = T$, and note that, although h_0 is zero for $x \geq \lambda t^{1/3}$, we expect that $\bar{h}_1$ will not be zero at this point and write $\ell(t) = \lambda t^{1/3} + \epsilon\ell_1(t)$ where $\ell_1 > 0$. With these assumptions, the leading order problem in the boundary layer may be formulated. Setting $\bar{h}_1 = \phi(X,T)/T^{2/3}$, integrating once with respect to X, noting that T enters the equation only as a parameter, and choosing the constant of integration to be zero to satisfy mass conservation, we find that ϕ satisfies

$$-\frac{\lambda}{3} = -\frac{\lambda}{3}\mathcal{H}(X-\ell_1(T)) - \phi_X + \frac{\lambda}{3(\ell_1(T)-X)} + \left(\int_0^{2\ell/\epsilon} \frac{\partial\phi(S,T)}{\partial S}\frac{dS}{S-X}\right)_X \qquad (6)$$

or

$$\phi + \frac{\lambda}{3}(X-\ell_1(T))\mathcal{H}(X-\ell_1(T)) = 0 \qquad (7)$$

where $\mathcal{H}$ is the Heaviside unit step function. Since (7) merely asserts that the crack has compact support, we consider only (6). From the form of (6) it is apparent that another integration may be carried out. The constant of integration, denoted by K, will be determined from the matching condition.

The relevant constants may now be determined. Since the mass in the boundary layer is $O(\epsilon^2)$ (the boundary layer having height $O(\epsilon)$ and width $O(\epsilon)$, the constant A in (5) may be determined simply from mass conservation, giving

$$A = -\frac{\lambda}{9}(1 - 2\log 2)$$

Determining the two-term inner limit of the two-term outer solution, we find that for large X

$$\phi \sim \frac{\lambda}{3}\left(-\frac{2}{3} + \frac{\pi^2}{9} + \log\left(\frac{2\lambda T^{1/3}}{\epsilon X}\right)\right) + O(\epsilon\log\epsilon)$$

Moreover, a consideration of contributions to the integral term shows that the upper limit of the integral in (6) may be replaced by ∞. Using the matching condition to provide a relationship between K and ℓ_1 for given T and ϵ, ℓ_1 may be eliminated in favour of K. Further simplifying the problem by writing it in terms of the dependent variable $\theta(X)$ where

$$\theta(X) = K - \frac{\lambda}{3}\log(1+X) + \frac{\lambda X}{3} - \frac{\lambda}{3}(X-\ell_1)\mathcal{H}(X-\ell_1) - \phi(X),$$

we find that θ satisfies the linear singular integro-differential equation

$$0 = \theta + \frac{\lambda}{3}\log\left(\frac{1+X}{X}\right) + \frac{\lambda}{3(1+X)}\log X - \int_0^\infty \frac{\theta'(S)}{S-X}dS \qquad (8)$$

with

$$\theta(0) = K, \quad \theta \to 0 \quad \text{as} \quad X \to \infty.$$

Some discussion of (8) is in order. The parameter K must be determined, and, according to the stress singularity condition, we seek the solution that satisfies $\theta'(0) = 0$. Assuming that the equation and boundary conditions (8) possess a unique solution for any K, (though this assertion is, in itself, a non-trivial matter) it is plausible that there is precisely one K for which $\theta'(0) = 0$, though evidently this is not easy to show. The perturbation to the crack length is then recovered from the relationship

$$\ell_1 = -\frac{3K}{\lambda} + \frac{\pi^2}{9} - \frac{2}{3} + \log\left(\frac{2\lambda T^{1/3}}{\epsilon}\right).$$

It is not clear whether (8) may be solved in closed form. A general procedure for solving semi-infinite range linear singular integro-differential equations was developed in [7], but its application is limited to cases where the Laplace transform of $f(x)$ is a polynomial. In the present case, the Laplace transform of $f(x)$ involves exponential integral and Meijer G-functions and, in view of the fact that the asymptotic and numerical behaviour of the solution may easily be established, we do not pursue the matter of a closed-form solution further.

Numerically, the solution of (8) is a fairly standard exercise (see, for example [3]) It is convenient to transform the equation to a finite range by setting

$$X = \frac{1+x}{1-x}, \quad \theta(X) = \zeta(x),$$

so that the equation becomes

$$\zeta(x) - \int_{-1}^{1} \frac{(1-x)(1-s)\zeta'(x)}{2(s-x)} ds = -\frac{\lambda}{6}(1-x)\log\left(\frac{1+x}{1-x}\right) - \frac{\lambda}{3}\log\left(\frac{2}{1+x}\right)$$

with

$$\zeta(-1) = K, \quad \zeta(1) = 0$$

The asymptotic behaviour of ζ may be determined at the ends of the range without difficulty, and using piecewise linear approximations for ζ away from the vicinity of collocation points and quadratic approximations close to collocation points, the equation may be reduced in an obvious way to a system of simultaneous linear equations which may then be solved using a library routine Some simple numerical tests showed that the method performed well, and the solutions obtained agreed with the asymptotic estimates for ζ. One significant advantage of the boundary layer analysis that has been performed is that, to determine f_1 as a function of T and ϵ, (8) needs to be solved once only As an example, using 200 collocation points the value of K for which $\zeta'(-1) = 0$ was given by $K = -0.21$, so that the crack length is given by

$$f(t) = \lambda t^{1/3} + \epsilon\left[-\frac{0.72}{\lambda} + \frac{\pi^2}{9} - \frac{2}{3} + \log\left(\frac{2\lambda t^{1/3}}{\epsilon}\right)\right]$$

4. Conclusions

Once the crack profile has been determined using the method described above, the flow law (2) may be used to determined the pressure, whence (1) may be used to yield σ_{yy} All the relevant details of the crack opening process may therefore be established We note that the pressure (relative to the reference stress) is zero at the crack tips, and the pressure in the crack varies smoothly throughout the crack. This contrasts with the case when other crack or flow laws are used, (see [1] for details) and provides confirmation of the fact that constitutive laws that may, on the face of it, seem to be somewhat similar, may give rise to very different physical predictions The fact that the crack tip pressure remains bounded suggests that the model presented above does not require any modification in regions near the crack tip.

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5. References

1 FITT, A.D., KELLY, A.D., PLEASE, C P · Crack propagation models for rock fracture in a geothermal energy reservoir To appear, SIAM J Appl Math.
2 GOODMAN, R E.: The mechanical properties of joints. Proc 3rd Congr. ISRM, Denver, 1A (1947) 127-140.
3 IVANOV, V.V : The theory of approximate methods and their application to the numerical solution of singular integral equations. Noordhoff International, (1977)
4 KING, J.R., PLEASE, C P.: Diffusion of dopant in crystalline silicon: an asymptotic analysis. I.M.A. J. Appl. Math. 37 (1986), 185-197.
5 MUSKHELISHVILI, N.I.: Some basic problems in the mathematical theory of elasticity. Groningen, Leyden, Holland; P. Noordhoff N.V. (1953).
6 PINE, R.J., CUNDALL, P.A.: Applications of the fluid-rock interaction programme (FRIP) to the modelling of hot dry rock geothermal energy systems. Proc. Int. S Fundamentals, (September 1985), 293-302.
7 VARLEY, E., WALKER, J.D.A.: A method for solving singular integrodifferential equations. I M.A. J. Appl. Math. 43 (1989), 11-45

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