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A Mathematical Model for Tonometry

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Summary. A mathematical model which describes the functioning of a Goldmann-type applanation tonometer is proposed in order to verify the validity of the Imbert-Fick principle. The spherical axisymmetric elastic equilibrium equations are solved using a Love Stress function. Conclusions are drawn regarding the circumstances under which the Imbert-Fick principle may or may not be valid

1 Introduction

Ophthalmologists define the intraocular pressure (IOP) to be the difference between the pressure inside a human eye and atmospheric pressure. A normal IOP lies in the range 12–25 mmHg; abnormally high IOP (glaucoma) is a serious condition that may lead rapidly to blindness. It is therefore vital to be able to carry out quick and reliable measurements of a patient's IOP. A widely-used instruments for this purpose is the Goldmann tonometer, which consists in essence of a flat circular tip connected to a spring. The tip is placed on the centre of the cornea of a topically anaesthetised eye and pushed towards it by adjusting the spring until the area of the flattened cornea is equal to the area of the circular tip. To determine the IOP, the "Imbert-Fick principle" is invoked. This states that if the diameter of the tonometer tip is exactly 3.06 mm, then each 0.1 gm force required to produce the flattening ("applanation") corresponds to 1 mmHg of intraocular pressure (see [2]).

Although the Goldmann tonometer is widely trusted by clinical practitioners, The Imbert-Fick principle is over 100 years old and is the result neither of detailed mathematical calculations nor even of an engineering correlation. It assumes that the eye is a thin, flexible hollow sphere and many have long suspected that, while it is often accurate, it may lead to inaccuracies for either eyes with abnormally high IOPs (see [7]) or that are subject to surgery (for example, scleral buckling). Our aim is thus to propose a mathematical model for the tonometry procedure to assess the validity the Imbert-Fick principle.

2 Mathematical Modelling

To develop a mathematical model for tonometry, we consider the eye to be linearly elastic hollow sphere subject to atmospheric pressure P_{at} at its outer surface $r = a$ and a pressure equal to $P_{at} + IOP$ at its internal surface $r = b$ (see Fig 1). The additional tonometer pressure T is assumed to act over the region $-\alpha \leq \theta \leq \alpha$ and we assume axisymmetry

The geometry and the axial symmetry of the problem give the following equations of elastic equilibrium (in spherical coordinates (r, θ, ϕ))

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{r\theta} + \frac{1}{r}[2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \cot\theta\sigma_{r\theta}] = F_r \ , \tag{1}$$

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta\theta} + \frac{1}{r}[3\sigma_{r\theta} + \cot\theta(\sigma_{\theta\theta} - \sigma_{\phi\phi})] = F_\theta \tag{2}$$

Here σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{\phi\phi}$ and $\sigma_{r\theta}$ denote the components of the stress tensor while F_r and F_θ represent the components of any equilibrating force required.

We now consider the boundary conditions. In this study we assume, for simplicity, that the normal and shear stresses are prescribed at both $r = a$ and $r = b$. We also assume that the equilibrating forces F_r and F_θ in (1) and (2) are supplied by the eye socket. Therefore

$$\sigma_{r\theta}(a) = 0, \quad \sigma_{rr}(a) = -G(\theta)$$
$$\sigma_{r\theta}(b) = 0, \quad \sigma_{rr}(b) = -P_{int} = -P_{at} - IOP \ ,$$

where

$$G(\theta) = \begin{cases} P_{at} + T & 0 \leqslant \theta \leqslant \alpha \\ p_{at} & \alpha < \theta \leqslant \pi \end{cases},$$

and T denotes the pressure exerted by the tonometer over the contact region.

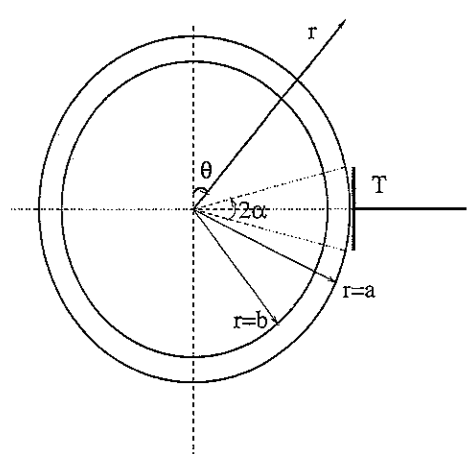

Fig. 1. Mathematical representation of the tonometer problem

2.1 Model Solution

Although the theory of the Airy stress function for two-dimensional elastic problems is well known, it seems to be less commonly appreciated that a corresponding "Love stress function" exists for axisymmetric spherical problems. Space permits few details, but the key result is that the homogeneous versions of (1)– (2) are automatically satisfied when u_r, u_θ, σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ are defined in terms of a biharmonic function $\chi(r,\theta)$ by the expressions in given in [1] This reduces the problem to the determination of a biharmonic Love stress function $\chi(r,\theta)$ that satisfies the correct boundary conditions. Using the general form of a separable biharmonic function, we assume that

$$\chi(r,\theta) \;=\; \sum_{n=0}^{\infty}\Big(A_n r^{-n-1} + B_n r^{-n+1} + C_n r^n + D_n r^{n+2}\Big)P_n(\cos\,\theta)\;,$$

where A_n, B_n, C_n, D_n are constants and $P_n(\cos\,\theta)$ denotes the n^{th} degree Legendre polynomial. Expressions for σ_{rr} and $\sigma_{r\theta}$ may now be obtained and the boundary conditions may be applied. After a great deal of algebraic manipulation (MAPLE was used to simplify the calculations) a system of coupled equations for the A_n, B_n, C_n and D_n may be derived With some effort this system may be solved and the displacements and stresses found; for brevity we do not show these expressions, which are very long and involved.

3 Results and Conclusions

The accuracy of the model depends on the parameters used. The typical diameter of a human eye is 25 mm ([6]) while the average thickness of the central cornea is 0 52 mm ([5], [4]) giving $a = 0\ 0125$ mm and $b = 0.01198$ mm. For a Goldmann tonometer $\alpha = \frac{\pi}{25}$. We took Poisson's ratio to be $\nu = 0\ 49$, and used a modulus of elasticity $E = 0\ 0229\ IOP$ proportional to the IOP ([5])

The qualitative behaviour of the model when the pressure T exerted by the tonometer takes different values for the same IOP is shown in Fig. 2 A normal IOP of 15 mmHg was used. The figure shows how the external wall of the eye is deformed when values of $T = 0\ 5\ IOP$, $T = 1\ 0\ IOP$ and $T = 2\ 0\ IOP$ are applied. We note that when the pressure exerted by the tonometer is equal to half of the IOP, the outer wall of the eye is barely deformed; when both are equal, a flattened region is observed and when the pressure exerted by the tonometer is two times greater than the IOP, a slightly re-entrant indentation is present. These results suggest that the Imbert-Fick law is qualitatively valid.

As far as quantitative results are concerned, if both the mathematical model and the Imbert-Fick principle were exactly in agreement, then one would expect that, for any IOP, the same amount of corneal flattening should be produced when pressure applied by the tonometer is equal to the IOP Calculations made using the model show, however, that this

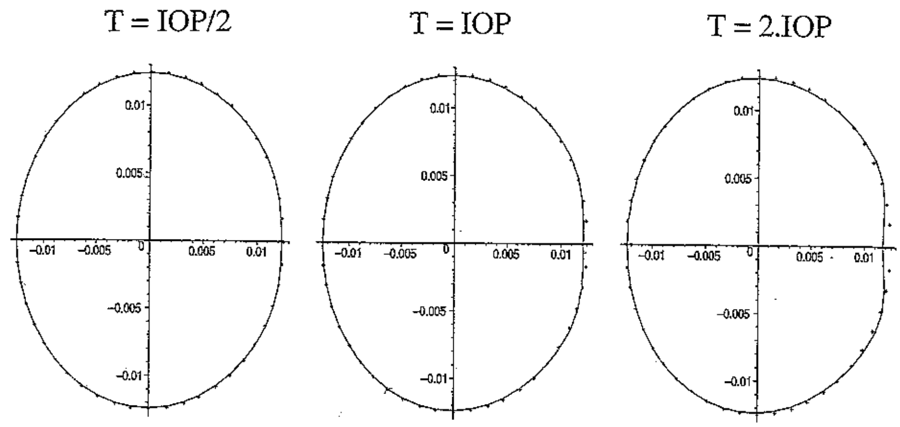

Fig. 2. Qualitative behaviour of the tonometer at different values of T for the same IOP (*Solid line*: deformed surface. *Dots*: undeformed surface)

is not exactly the case. In particular, for IOPs of about 40 mmHg and greater, a re-entrant indentation is present. This suggests either that the model has shortcomings (see below for further discussion) or that the Imbert-Fick principle is flawed (at least for large IOPs).

It is known from experimental studies that the Imbert-Fick law can give inaccurate results for elevated IOPs. The left-hand diagram of Fig. 3 shows the amount of flattened area vs the ratio between the flattening force and IOP for both the Imbert-Fick law and the experimental results in [3], confirming that the Imbert-Fick law is not universally applicable. The right-hand diagram of Fig. 3 shows detailed results obtained from the mathematical model for different values of the IOP. We observe that the amount of flattened area varies not only with the ratio of the applanating

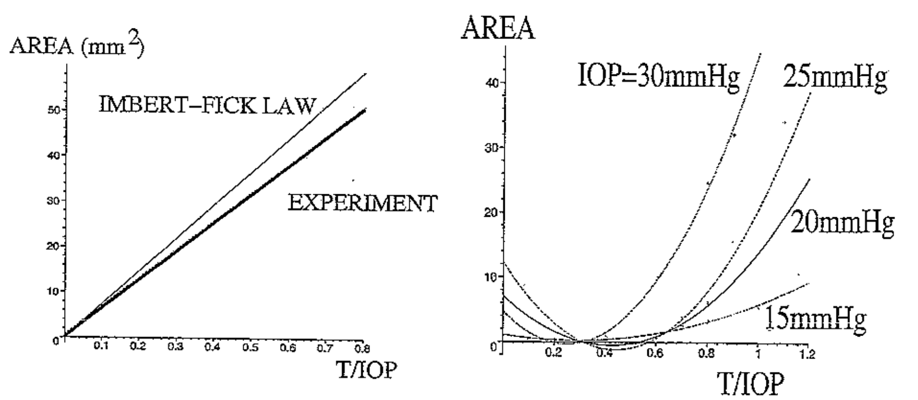

Fig. 3. (*Left-hand diagram*) Imbert-Fick law compared to experimental results, (*Right-hand diagram*) Model results for various IOPs

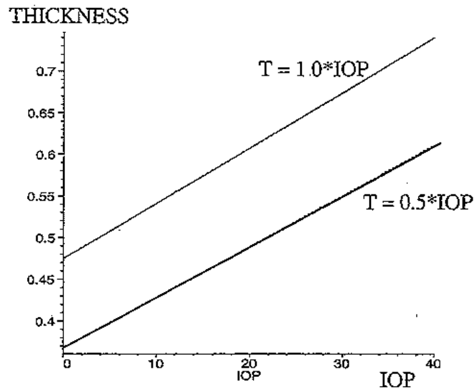

Fig. 4. Relationship between corneal thickness and IOP required for Imbert-Fick law to be correct

force to the IOP but also displays a dependence on the value of IOP. We conclude that both the Imbert-Fick law and the mathematical model require reconsideration.

One solution to the problems that have arisen in the mathematical model is to assume that the IOP is related to the corneal thickness Fig. 4 shows the required relationship for various different tonometer pressures ($T = 1.0\ IOP$ and $T = 0.5\ IOP$). It can be seen that when $T = 1.0\ IOP$, then for any value of the IOP a specific corneal thickness exists that renders the tonometer reading "correct". Thus the tonometer may be used to measure the IOP accurately provided it is modified by a correction factor K ($IOP = T/K$), where K is determined by both the IOP read by the tonometer and the thickness of the cornea.

In conclusion, a mathematical model of an applanation tonometer has been proposed. Initial results suggest that the Imbert-Fick law is qualitatively valid; For a quantitative point of view, however, the model requires some modification, especially at elevated IOPs. We therefore conclude that neither our model nor the Imbert-Fick principle can be trusted in every circumstance. One possibility to correct the model is to take account of the corneal or scleral thickness, and this provides a simple practical correction to the model inaccuracies. In reality, however, the results tell us (a) that the Imbert-Fick principle is limited in its applicability and (b) that we are probably solving the wrong elastic problem. Since the tonometer functions precisely by flattening a particular region of the cornea, we should solve a mixed boundary value problem, prescribing the appropriate radial displacement in the region of contact between the tonometer and the eye and zero radial stress everywhere else on the outer surface of the eye. Mathematically, this problem is a good deal harder to solve, and is the focus of a current study.

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Modelling
Reservoir

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Abstract. The
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