A HIGH-REYNOLDS-NUMBER CROSS-FLOW WITH INJECTION AND SUCTION

By A. D. FITT *and* T. R. B. LATTIMER

(*Faculty of Mathematical Studies, University of Southampton, Highfield, Southampton* SO17 1BJ)

[Received 11 January 1996. Revise 15 March 1996]

SUMMARY

The effect upon a high-Reynolds-number cross-flow of an upstream injection slot and a downstream suction slot of given geometries and strengths is examined. It is shown that the problem may be reduced to a single nonlinear singular integrodifferential equation. It transpires that, in the resulting flow, a total of five different regimes may be identified. For *critical* suction, the suction strength is just sufficient to reingest all of the previously injected flow. For weaker suctions, the flow is either *subcritical*, in which case the injected flow that cannot be reingested forms a layer downstream of the suction slot, or *subsubcritical*, in which case the low-pressure region produced by the injection is of sufficient strength that the 'suction' slot exhausts, rather than ingests, fluid. For suctions stronger than the critical value, the flow is *supercritical*, and the suction slot ingests some of the cross-flow as well as the previously injected flow, leading to an order-of-magnitude increase in the mass flow into the slot. Finally, for *supersupercritical* flow, when the suction strength is an order of magnitude larger than in cases previously considered, the injection slot is effectively absent and the mass flow into the slot once again jumps by an order of magnitude. In each case the equation governing the flow is solved asymptotically and numerically. Some limiting cases are also identified in which closed-form solutions may be determined.

1. Introduction

In this study the problem of a two-dimensional high-Reynolds-number cross-flow over a flat plate is considered in the case where the plate contains two slots. The pressures in the upstream and downstream slots are respectively just greater than and just less than the free-stream static pressure, so that the upstream slot injects a small quantity of fluid into the cross-flow, whilst the downstream slot gives rise to a suction from the flow.

The original motivation for considering this problem arose from a study of pollutant release from an exhaust channel. Under normal circumstances, no pollutant can escape from the channel. In exceptional cases, however, the pressure in the channel increases and small amounts of pollutant may be released. Under these conditions, the existence of a downstream suction slot allows escaping pollutant to be recaptured and effectively removed from the cross-flow. Since, from a practical point of view, the parameters known in the problem are likely to be the pressures in the slots, the main

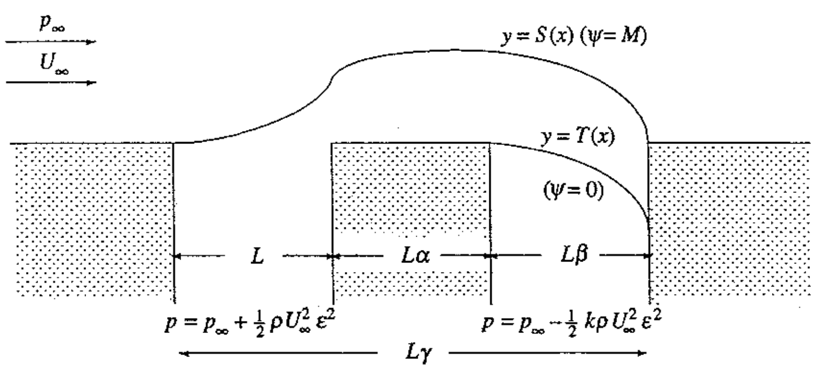

FIG 1 Schematic diagram of slot injection/suction geometry for the critical suction case

quantities of interest are the mass flows out of and into the respective slots. Understanding how these vary as functions of the slot pressures and geometry will allow the design of the pollutant removal system to be optimized.

As well as pollutant removal, there are many other physical applications where a combination of injection and suction may be important. It was shown in (**1**) that boundary-layer separation and heat transfer could be materially affected by injection, whilst the combined effects of injection and suction have long been studied in relation to other forms of boundary layer control (especially in relation to the laminarization of flow over wing sections), the design of jet flaps, and the maximization of wake reduction for projectiles.

A schematic diagram of the geometry under consideration is given in Fig. 1. The injection slot width is denoted by L, whilst the distance from the downstream edge of the injection slot to the upstream edge of the suction slot is given by $L\alpha$. The width of the suction slot is denoted by $L\beta$. The streamlines are labelled (with non-dimensional variables; for details see below) for the critical suction case described below.

2. The critical suction case

In this section, we consider the *critical* suction case where, for a given geometry, the suction parameter k takes a value which ensures that all of the injected flow and none of the free-stream flow is sucked back into the suction slot. As far as pollution removal is concerned, therefore, this may be regarded as the optimal suction arrangement since the maximum amount of pollutant is retrieved for a minimum pressure requirement.

It is shown below that the problem of determining the flow for the slot arrangement shown in Fig. 1 may be reduced to that of solving a single nonlinear singular integrodifferential equation. The analysis is similar to that given in (**2**), which itself has its origins in the ideas contained in the 'inviscid boundary layer' study of (**3**). Since the determination of the flow for other suction strengths will be similar, the model presented here will require only minor alterations to treat cases where the suction is non-critical. The full viscous interaction problem for plate injection into a separated supersonic boundary layer was studied by Smith and Stewartson in (**4**). They found that, for blowing of order $U_\infty \mathrm{Re}^{-\frac{3}{8}}$, a region of separated flow of width $L\,\mathrm{Re}^{-\frac{3}{8}}$ is formed upstream of the slot. To lowest order, however, this region will not affect the details of injection and suction and may be ignored.

The flow is assumed to be inviscid, incompressible and irrotational. Far away from the slots, the free stream is assumed to have a pressure p_∞ and a velocity U_∞. We take the injection pressure in the upstream (injection) slot to be $p_\infty + \frac{1}{2}\rho U_\infty^2 \varepsilon^2$, thus defining the small parameter ε in the problem; the pressure in the downstream suction slot is given by $p_\infty - \frac{1}{2}k\rho U_\infty^2 \varepsilon^2$ where $k > 0$. The slot pressures may therefore be controlled independently of each other (changing the injection pressure amounts merely to a redefinition of the small parameter ε). Since the injection-slot pressure is greater than the free-stream pressure only by a small amount, the injection flow forms a thin layer which, using a well-known result of thin aerofoil theory (see, for example, (**5**)), may be expected to have a height of order $\varepsilon^2 L$.

Non-dimensionalizing velocities and distances in the outer flow with U_∞ and L respectively, the non-dimensional stream function for the outer flow is given, by the standard thin aerofoil source-distribution model (see, for example, (**6**)), by

$$\psi = y + \frac{\varepsilon^2}{\pi}\int_0^\infty S'(t)\tan^{-1}\left(\frac{y}{x-t}\right)dt + o(\varepsilon^2).$$

The term $S'(t)$ in the integrand arises from the fact that S has been non-dimensionalized with $L\varepsilon^2$, and ψ has been forced to satisfy the kinematic boundary condition on $y = S(x)$. Using Bernoulli's equation, the corresponding non-dimensional pressure in the outer flow is therefore given by

$$p = \frac{p_\infty}{\rho U_\infty^2} - \frac{\varepsilon^2}{\pi}⨍_0^\infty \frac{S'(t)}{x-t}\,dt, \tag{1}$$

where, as usual, the bar on the integral sign denotes a Cauchy principal value integral.

We now consider the flow over the slots and the flow between the slots separately. The order of magnitude of the mass flow from the injection slot may be determined by noting that, for the pressure perturbations to be within $O(\varepsilon^2)$ of p_∞, as suggested by (1), the velocity in the injected layer downstream of the injection slot must be $O(\varepsilon U_\infty)$. Since this layer has a thickness of order $\varepsilon^2 L$, the mass flow must be $O(\varepsilon^3 \rho L U_\infty)$. We therefore define the mass flow by $M\varepsilon^3 \rho L U_\infty$, where M is an $O(1)$ constant to be determined. A consequence of these orders of magnitude is that a vortex sheet separates the injected and outer flows.

Deep within the injection slot the flow is assumed to be uniform. From the mass-flow result given above, we conclude that velocities here must therefore be of order $\varepsilon^3 U_\infty$, so that the non-dimensional total slot pressure is given by

$$p = \frac{p_\infty}{\rho U_\infty^2} + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^6). \tag{2}$$

We assume henceforth that these orders of magnitude apply all the way upstream to the vortex sheet over the top of the slot. The physical basis for this important assumption lies in the fact that, since the total pressure in the free-stream flow is greater than that of the injected flow, the free stream acts as a 'lid' which effectively allows flow to escape from the slot only at the downstream corner. (For further discussion and experimental evidence for this effect, see (**2**).)

In the region $1 < x < 1 + \alpha$ between the slots, we scale x with L, y and S with $\varepsilon^3 L$, ψ with $\varepsilon^2 U_\infty L$ and p with ρU_∞^2. The scaled Laplace equation for ψ becomes

$$\psi_{yy} = 0$$

to leading order, and, since by the definition of the mass flow it must be the case that

$$\psi(x, 0) = 0 \quad \text{and} \quad \psi\big(x, S(x)\big) = M,$$

we find that, in this region,

$$\psi = My/S(x).$$

The non-dimensional pressure between the slots may now be determined by applying Bernoulli's equation, yielding

$$p = \frac{p_\infty}{\rho U_\infty^2} + \tfrac{1}{2}\varepsilon^2\left(1 - \frac{M^2}{S^2}\right). \tag{3}$$

In and above the downstream suction slot $1 + \alpha < x < \gamma$ (where $\gamma = 1 + \alpha + \beta$) the injected flow still forms a thin layer, so that the non-dimensional pressure can be a function only of x to leading order, and is thus given by the constant

$$p = \frac{p_\infty}{\rho U_\infty^2} - \tfrac{1}{2}k\varepsilon^2. \tag{4}$$

One consequence of this is that the streamline that emanates from $(1 + \alpha, 0)$ attaches to the downstream suction slot wall. This reflects the fact that an inner region exists within a distance of $O(\varepsilon L)$ of the downstream suction slot wall, though analysis of this inner region is not required in order to predict the mass flow into the suction slot.

For the theory described above to be valid, the slot width L must be small compared to any other physical dimension in the flow. For the injected flow to be essentially inviscid, the Reynolds number $LU_\infty/\nu \gg 1$ and the injection strength ε must satisfy $\mathrm{Re}^{-\frac{1}{3}} \ll \varepsilon$. The injection must also be sufficiently strong to blow off the viscous boundary layer. This will certainly be the case if ε exceeds $O(\mathrm{Re}^{-\frac{1}{6}})$; in some circumstances, however, boundary-layer blow off may still occur for smaller values of ε since towards the trailing edge of the slot the vertical velocity v may reach $O(U_\infty \varepsilon^2)$.

Having effectively determined the flow in each region, all that remains is to ensure that the pressure is continuous across the dividing streamline $y = S(x)$. This may be achieved simply by comparing (1), (2), (3) and (4) to finally retrieve the nondimensional equation

$$\frac{1}{\pi}\int_0^\gamma \frac{S'(t)}{t-x}\,dt = \begin{cases} \frac{1}{2} & (0 \leqslant x < 1), \\ \dfrac{1}{2} - \dfrac{1}{2}\dfrac{M^2}{S^2} & (1 < x < 1+\alpha), \\ -\frac{1}{2}k & (1+\alpha \leqslant x < \gamma) \end{cases} \tag{5}$$

The equation (5) is a nonlinear singular integrodifferential equation, which must be solved subject to the boundary conditions $S(0) = S'(0) = 0$. The suction strength k and the mass flow M emerging from the slot are unknown for a given geometry and must be determined as part of the solution. A final boundary condition is given by the requirement that since, in this case, the injected flow is completely absorbed by the suction slot, we must have $S(\gamma) = 0$. It is worth pointing out that, although the pressure is unknown (and indeed discontinuous) at $x = 1$, subsequent inversions of (5) are insensitive to this and do not affect the model.

For *linear* singular integrodifferential equations on semi-infinite domains, it was shown in (7) that in many cases a closed-form solution could be determined. When the range is finite, as in this case, it seems that few exact solutions are available. In the present case, the nonlinear nature of (5) renders it unlikely that closed-form solutions to the problem are possible (though see the next section for a particular case). Notwithstanding this, the important properties of $S(x)$ may be determined asymptotically without too much trouble. Using standard methods, we find that

$$S(x) \sim K_1 x^{\frac{3}{2}} \quad (x \sim 0), \tag{6}$$

$$S(x) \sim S(1) + K_2^{\pm}\,|1-x|\log|1-x| \quad (x \sim 1\pm), \tag{7}$$

$$S(x) \sim K_3(\gamma - x)^{\frac{1}{2}} \quad (x \sim \gamma). \tag{8}$$

As is common in such problems, although expressions for the constants K_1, $K_2^{\pm}$ and K_3 may be written down, they are global in that they involve integrals of S over the whole range and therefore cannot be determined a priori.

In practice, the main item of concern for a particular given geometry is the value taken by k and the corresponding mass flow M, and to determine these parameters it is necessary to solve (5) numerically. The form of the equation given above is awkward as it contains both singular integrals and derivatives. For numerical purposes the equation must therefore be reformulated. We begin by inverting (5) on the interval $[0, \gamma]$ using standard singular integral-equation methods (see, for example (**8**) or (**9**)). Choosing the inversion and the eigenfunction involved therein to ensure that $S'(0) = 0$ gives

$$S'(x) = \frac{M^2}{2\pi}\left(\frac{x}{\gamma - x}\right)^{\frac{1}{2}} \int_1^{1+\alpha} \left\{\frac{\gamma - t}{t}\right\}^{\frac{1}{2}} \frac{dt}{S^2(t)(t - x)}$$
$$+ \left\{\frac{x}{\gamma - x}\right\}^{\frac{1}{2}} \left[\frac{1-k}{4} + \left(\frac{1+k}{2\pi}\right)\sin^{-1}\left(\frac{2 + 2\alpha - \gamma}{\gamma}\right)\right]$$
$$+ \left(\frac{1+k}{2\pi}\right)\log\left(\frac{(\{(1+\alpha)(\gamma - x)\}^{\frac{1}{2}} + \{x\beta\}^{\frac{1}{2}})^2}{\gamma\,|1 + \alpha - x|}\right)$$

If this expression is then integrated between 0 and x, the resulting Fredholm integral equation contains neither derivatives nor Cauchy principal value integrals and automatically satisfies $S(0) = 0$. Carrying out this integration gives

$$S(x) = \frac{M^2}{2\pi}\int_1^{1+\alpha} \frac{f_1(x, t)}{S^2(t)}\,dt + g_1(x), \tag{9}$$

where

$$f_1(x, t) = \left\{\frac{\gamma - t}{t}\right\}^{\frac{1}{2}}\left[-\tfrac{1}{2}\pi - \sin^{-1}\left(\frac{2x - \gamma}{\gamma}\right)\right]$$
$$+ \log\left(\frac{(\{x(\gamma - t)\}^{\frac{1}{2}} + \{t(\gamma - x)\}^{\frac{1}{2}})^2}{\gamma\,|t - x|}\right),$$

$$g_1(x) = \left[\frac{1-k}{4} + \left(\frac{1+k}{2\pi}\right)\sin^{-1}\left(\frac{2 + 2\alpha - \gamma}{\gamma}\right)\right]$$
$$\times\left[\frac{\pi\gamma}{4} + \frac{\gamma}{2}\sin^{-1}\left(\frac{2x - \gamma}{\gamma}\right) - \{x(\gamma - x)\}^{\frac{1}{2}}\right]$$
$$+ \left(\frac{1+k}{2\pi}\right)\left[(x - 1 - \alpha)\log\left(\frac{(\{(1+\alpha)(\gamma - x)\}^{\frac{1}{2}} + \{x(\gamma - 1 - \alpha)\}^{\frac{1}{2}})^2}{\gamma\,|1 + \alpha - x|}\right)\right.$$
$$\left. + \{(1+\alpha)(\gamma - 1 - \alpha)\}^{\frac{1}{2}}\left(\tfrac{1}{2}\pi + \sin^{-1}\left(\frac{2x - \gamma}{\gamma}\right)\right)\right].$$

Numerical calculations may now be performed using a direct iteration relaxation method. Assuming that $S(x)$ is piecewise constant on each interval $[x_j, x_{j+1})$, $(x_0 = 1, \ldots, x_N = 1 + \alpha)$, the integrals in (9) may be performed analytically to give the iterative relaxation scheme

$$\bar{S}^{(i+1)}(x_j) = \sum_{k=0}^{N-1} (S^{(i)}(t_k))^{-2}[f_3(x_j, t_{k+1}) - f_3(x_j, t_k)] + g_1(x_j),$$
$$S^{(i+1)}(x_j) = S^{(i)}(x_j) + \varepsilon_r(\bar{S}^{(i+1)}(x_j) - S^{(i)}(x_j)),$$

where

$$f_3(x, t) = \frac{M^2}{2\pi}\left[\left(\frac{\gamma}{2}\sin^{-1}\left(\frac{2t-\gamma}{\gamma}\right) + \{t(\gamma - t)\}^{\frac{1}{2}}\right)\left(-\tfrac{1}{2}\pi - \sin^{-1}\left(\frac{2x-\gamma}{\gamma}\right)\right)\right.$$
$$+ \{x(\gamma - x)\}^{\frac{1}{2}}\sin^{-1}\left(\frac{2t-\gamma}{\gamma}\right)$$
$$\left. + (t - x)\log\left(\frac{(\{x(\gamma - t)\}^{\frac{1}{2}} + \{t(\gamma - x)\}^{\frac{1}{2}})^2}{|t - x|}\right) - t\log\gamma\right].$$

This scheme is to be applied for $i > 0$, $j = 0, 1, \ldots, N$, with $S^{(0)}(x_j)$ 'guessed'. (In practice, the method proves insensitive to the values taken for $S^{(0)}$. In all the examples considered here and described below, initial values of $S^{(0)}(x) \equiv 1$ were used.) When convergence is achieved, the above formula may be applied for arbitrary values of x in order to determine the solution for $x \in [0, 1] \cup [1 + \alpha, \gamma]$. In practice, it was found that in most cases a value of about 0·3 for the relaxation parameter ε_r was sufficient to allow fast convergence.

The numerical scheme described above allows the equation to be solved for given values of k and M. In general, however, the solution obtained will fail to satisfy the boundary condition $S(\gamma) = 0$. In order to compute solutions that satisfy this condition, some extra consideration of the problem is necessary. Considering the region directly over the suction slot (as shown in Fig. 1) we note that the dimensionless scaled equation $\psi_{yy} = 0$ must be solved to determine $T(x)$ to leading order. This yields

$$\psi = \frac{M(y - T)}{S - T}.$$

Following a streamline in the region $T(x) < y < S(x)$ back to the injection slot, we find that, since the dimensionless pressure over the slot is $p_\infty/\rho U_\infty^2 - \frac{1}{2}k\varepsilon^2$, Bernoulli's equation gives

$$\frac{p_\infty}{\rho U_\infty^2} + \tfrac{1}{2}\varepsilon^2 = \frac{p_\infty}{\rho U_\infty^2} - \tfrac{1}{2}k\varepsilon^2 + \frac{M^2\varepsilon^2}{2(S - T)^2}$$

and hence

$$S(x) - T(x) = M/\{1 + k\}^{\frac{1}{2}}. \tag{10}$$

This result shows that, over the suction-slot region, the height $S - T$ is constant, and, since for the critical case necessarily $S(\gamma) = 0$, it follows that the streamline $T(x)$ reattaches to the downstream wall of the suction slot, the flow beneath $T(x)$ being essentially stagnant. It should be noted that, were the full problem to be solved, the streamline $T(x)$ would not be expected to attach in this manner. However, since the asymptotic analysis presented above is clearly not valid within a distance $O(\varepsilon)$ from the downstream suction-slot wall, the fact that $T(\gamma)$ is finite is a consistent result.

Equation (10), being essentially a mass-flow condition, also allows unique values of k and M to be determined for a given geometry as follows: for fixed k, the value of M is determined for which $S(\gamma)=0$. Evaluating (10) at $x=1+\alpha$ shows that, since $T(1+\alpha)=0$,

$$M=S(1+\alpha)(1+k)^{\frac{1}{2}}. \tag{11}$$

Using (11) to determine a more accurate update of M, an iterative procedure may easily be constructed that allows rapid convergence to the unique values of k and M that lead to solutions where both of the conditions $S(\gamma)=0$ and (11) are satisfied.

Figure 2 shows flow streamlines for a typical critical case with $\alpha=\beta=1$. In all computations described below (unless otherwise stated) 100 mesh points were used, and the solution was deemed to have converged when $|S^{(i+1)}(x_j)-S^{(i)}(x_j)|<10^{-6}$ for all x_j. Computations (coded in FORTRAN) were carried out on a Sun Sparc-2, each one taking only a few CPU seconds to converge. For this geometry the values of k and M were 1·9146 and 1·9176 respectively. For clarity, a value of $\varepsilon=0{\cdot}3$ has been used here and in the other streamline diagrams below.

Figure 3 shows the values taken by k and M for a number of different critical suction geometries. In the first case α took a fixed value of 1, whilst the suction slot width β was varied. In the second set of results shown the process was reversed so that β took the fixed value 1 and α was varied. The results indicate that as the slot width or slot separation decreases k and M increase rapidly. The slow decrease in the critical value of k as β increases shows that, as might be expected, large suction slots are not particularly advantageous.

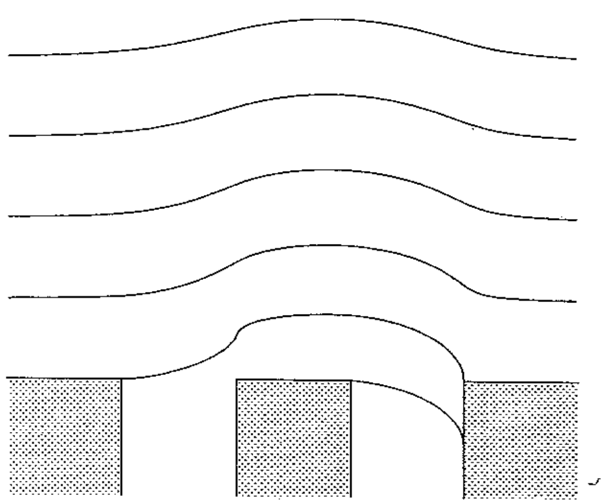

FIG. 2. Streamlines for the critical suction case with $\alpha=1$, $\beta=1$ and $M=1{\cdot}9176$, $k=1{\cdot}9146$

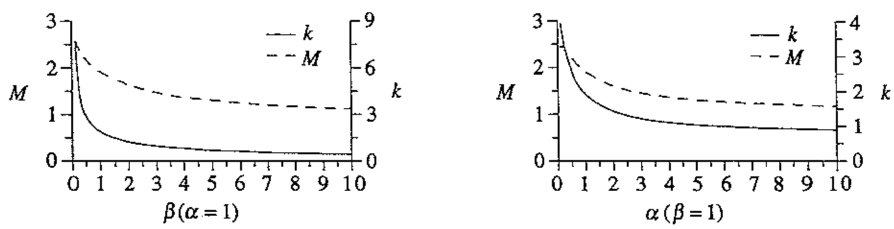

FIG. 3. Values of mass flow M and critical suction k for various slot geometries

2.1. *A special case of critical suction*

It is instructive to discuss a special case of the critical suction problem where a closed-form solution may be determined. If we take $\alpha = 0$, so that the suction slot begins where the injection slot ends, the problem becomes

$$\frac{1}{\pi}\int_0^\gamma \frac{S'(t)}{t-x}\,dt = \begin{cases} \frac{1}{2} & (0 \leq x < 1), \\ -\frac{1}{2}k & (1 < x < \gamma). \end{cases} \tag{12}$$

Inverting (12) as described above, we find that

$$S'(x) = \left(\frac{x}{\gamma - x}\right)^{\frac{1}{2}}\left[\frac{1-k}{4} + \left(\frac{1+k}{2\pi}\right)\sin^{-1}\left(\frac{2-\gamma}{\gamma}\right)\right]$$
$$+ \left(\frac{1+k}{2\pi}\right)\log\left(\frac{(\{\gamma - x\}^{\frac{1}{2}} + \{x(\gamma-1)\}^{\frac{1}{2}})^2}{\gamma\,|1-x|}\right)$$

As usual, the inversion and the eigenfunction constant have been chosen so that $S'(0) = 0$. This expression confirms that, as expected, $S'(x) \sim (\gamma - x)^{-\frac{1}{2}}$ as $x \to \gamma$. The solution may now be completed by integrating with respect to x from 0 to x (since $S(0) = 0$) to give

$$S(x) = \left[\frac{1-k}{4} + \left(\frac{1+k}{2\pi}\right)\sin^{-1}\left(\frac{2-\gamma}{\gamma}\right)\right]$$
$$\times\left[\frac{\pi\gamma}{4} + \frac{\gamma}{2}\sin^{-1}\left(\frac{2x-\gamma}{\gamma}\right) - \{x(\gamma - x)\}^{\frac{1}{2}}\right]$$
$$+ \left(\frac{1+k}{2\pi}\right)\left[\tfrac{1}{2}\pi\{\gamma - 1\}^{\frac{1}{2}} + \{\gamma - 1\}^{\frac{1}{2}}\sin^{-1}\left(\frac{2x-\gamma}{\gamma}\right)\right.$$
$$\left. + (x-1)\log\left(\frac{(\{\gamma - x\}^{\frac{1}{2}} + \{x(\gamma - 1)\}^{\frac{1}{2}})^2}{\gamma\,|1-x|}\right)\right].$$

For critical suction, this solution must be made to satisfy $S(\gamma) = 0$. This condition uniquely determines k, the critical suction, in terms of γ according to

$$k = \frac{\gamma\pi + 2\gamma \sin^{-1}((2-\gamma)/\gamma) + 4(\gamma - 1)^{\frac{1}{2}}}{\gamma\pi - 2\gamma \sin^{-1}((2-\gamma)/\gamma) - 4(\gamma - 1)^{\frac{1}{2}}}.$$

As far as the mass flow is concerned, the relationship (11) still applies, so that $M = (1 + k)^{\frac{1}{2}} S(1)$, thereby uniquely determining the mass flow for this critical suction case. In particular, we find that, when the width of the suction slot is identical to the injection slot so that $\gamma = 2$, then k and M are given by

$$k = \frac{2\pi + 4}{2\pi - 4}, \qquad M = \left(\frac{2\pi}{(\pi - 2)^3}\right)^{\frac{1}{2}}.$$

Using the formulae given above, the critical suction strength and the corresponding mass flow may be calculated for limiting values of the suction-slot width. For suction-slot widths that are large compared to the injection-slot width we find that

$$k \sim \frac{4}{\pi\sqrt{\gamma}} + O(\gamma^{-1}), \quad M \sim \frac{1}{\pi} + O(\gamma^{-\frac{1}{2}}) \quad (\gamma \to \infty).$$

These results are in accordance with what might be expected; as the relative width of the suction slot increases the suction required decreases, but the mass flow that can be produced by the injection slot approaches a limiting value.

For small suction slots, the corresponding results are

$$k \sim \frac{3\pi}{4(\gamma - 1)^{\frac{3}{2}}} + O((\gamma - 1)^{-\frac{1}{2}}), \quad M \sim \frac{3\sqrt{(3\pi)}}{8(\gamma - 1)^{\frac{5}{4}}} + O((\gamma - 1)^{-\frac{1}{4}}) \quad (\gamma \sim 1).$$

The suction strength required to achieve critical flow thus increases without limit, as does the corresponding mass flow that can be produced by the injection slot. It is worth pointing out that this limit must be regarded as being of academic interest only, since many of the assumptions and orders of magnitude inherent in the modelling are violated.

3. Subcritical suction

In section 2 it was shown how to determine the flow for a given geometry when the suction strength is fixed at the critical value. Here, we discuss the flow when k is less than the critical value for a given geometry. Clearly when the suction strength is insufficient to suck all of the previously injected flow into the downstream slot, a layer of injected fluid will be formed downstream of the suction slot. We denote the dimensionless mass flow in

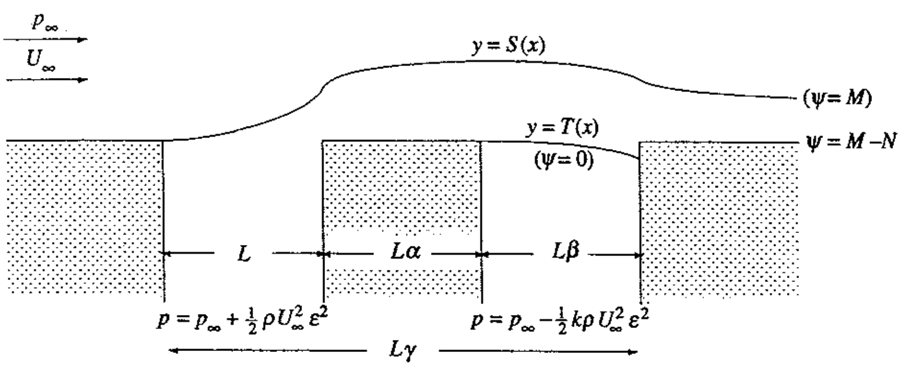

FIG. 4. Schematic diagram of slot injection/suction geometry for the subcritical suction case

this layer by N, and consider only the case where N is $O(\varepsilon^3 \rho L U_\infty)$. Other cases where the downstream layer has smaller order-of-magnitude thicknesses (and thus k is, for example, within $O(\varepsilon)$ of the critical) may also be considered, but, to leading order, these are merely the critical suction case. Labelling the (non-dimensional) streamlines as shown in Fig. 4, we note that, downstream of the suction slot, the scaled dimensionless stream function ψ is again determined, to leading order, by the equation $\psi_{yy} = 0$ and thus

$$\psi = \frac{Ny}{S} + M - N.$$

In this region, therefore, the non-dimensional pressure is given by

$$p = \frac{p_\infty}{\rho U_\infty^2} + \varepsilon^2\left(\frac{1}{2} - \frac{N^2}{2S^2}\right).$$

For subcritical suction, the integral equation that must be solved to determine the flow is thus given by

$$\frac{1}{\pi}\int_0^\infty \frac{S'(t)}{t-x}\,dt = \begin{cases} \frac{1}{2} & (0 \leq x < 1), \\ \frac{1}{2} - \frac{1}{2}\frac{M^2}{S^2} & (1 < x < 1+\alpha), \\ -\frac{k}{2} & (1+\alpha \leq x < \gamma), \\ \frac{1}{2} - \frac{1}{2}\frac{N^2}{S^2} & (\gamma < x < \infty) \end{cases} \tag{13}$$

This equation must be solved with $S(0) = S'(0) = 0$, and, for given k, both M and N must be determined. Once again, the asymptotic behaviour of $S(x)$ may easily be determined. Near $x = 0$ and $x = 1$, the behaviour is similar to that given by (6) and (7), whilst for large x it may be shown that

$$S(x) \sim N - \frac{N^2}{\pi x} + O\left(\frac{\log x}{x^2}\right) \quad (x \to \infty) \tag{14}$$

For numerical purposes, it is once again easiest to invert (13) to obtain

$$S'(x) = \frac{M^2\sqrt{x}}{2\pi} \int_1^{1+\alpha} \frac{dt}{\sqrt{t}S^2(t)(t-x)} + \frac{N^2\sqrt{x}}{2\pi} \int_\gamma^\infty \frac{dt}{\sqrt{t}S^2(t)(t-x)}$$
$$+ \left(\frac{1+k}{2\pi}\right) \log \left|\frac{(\sqrt{\gamma} - \sqrt{x})(\{1+\alpha\}^{\frac{1}{2}} + \sqrt{x})}{(\sqrt{\gamma} + \sqrt{x})(\{1+\alpha\}^{\frac{1}{2}} - \sqrt{x})}\right|$$

and then to integrate with respect to x. All singular integrals and derivatives are thus removed, the resulting expression being

$$S(x) = \frac{M^2}{2\pi} \int_1^{1+\alpha} \frac{1}{S^2(t)} f_2(x, t)\, dt + \frac{N^2}{2\pi} \int_\gamma^\infty \frac{1}{S^2(t)} f_2(x, t)\, dt$$
$$+ \left(\frac{1+k}{2\pi}\right)\Big[2\sqrt{x}(\{1+\alpha\}^{\frac{1}{2}} - \sqrt{\gamma})$$
$$+ (x - 1 - \alpha) \log \left(\frac{\sqrt{x} + \{1+\alpha\}^{\frac{1}{2}}}{|\sqrt{x} - \{1+\alpha\}^{\frac{1}{2}}|}\right) - (x - \gamma) \log \left(\frac{\sqrt{x} + \sqrt{\gamma}}{|\sqrt{x} - \sqrt{\gamma}|}\right)\Big],$$

where

$$f_2(x, t) = -2\sqrt{\frac{x}{t}} + \log \left(\frac{\sqrt{t} + \sqrt{x}}{|\sqrt{t} - \sqrt{x}|}\right)$$

The problem may now be solved by determining S in the regions $[1, 1 + \alpha]$ and $[\gamma, \delta]$, where δ denotes the position of the final numerical mesh point. In order to allow for the fact that, in practice, δ must clearly be finite, the tail of the infinite integral may be estimated asymptotically, giving

$$\frac{N^2}{2\pi} \int_\delta^\infty \frac{1}{S^2(t)} \left(-2\sqrt{\frac{x}{t}} + \log \left(\frac{\sqrt{t} + \sqrt{x}}{|\sqrt{t} - \sqrt{x}|}\right)\right) dt \sim \frac{N^2 x}{\pi S^2(\delta)} Q\left(\sqrt{\frac{\delta}{x}}\right) \quad (x \sim \infty),$$

where

$$Q(s) = s + \tfrac{1}{2}(1 - s^2) \log \left(\frac{s+1}{s-1}\right)$$

The final numerical scheme (with $x_0 = 1$, $x_{M_1} = 1 + \alpha$ and $x_{N_0} = \gamma$, $x_{N_1} = \delta$) thus becomes

$$\begin{aligned}\bar{S}^{(i+1)}(x_j) &= \frac{M^2}{2\pi}\sum_{k=0}^{M_1-1}(S^{(i)}(t_k))^{-2}[f_4(x_j, t_{k+1}) - f_4(x_j, t_k)]\\ &\quad + \frac{N^2}{2\pi}\sum_{k=N_0}^{N_1-1}(S^{(i)}(t_k))^{-2}[f_4(x_j, t_{k+1}) - f_4(x_j, t_k)]\\ &\quad + \frac{xN^2}{\pi S^2(\delta)}Q\left(\sqrt{\frac{\delta}{x_j}}\right) + g_4(x_j), \qquad (15)\\ S^{(i+1)}(x_j) &= S^{(i)}(x_j) + \varepsilon_r(\bar{S}^{(i+1)}(x_j) - S^{(i)}(x_j)),\end{aligned}$$

where

$$f_4(x, t) = -2\sqrt{(tx)} + (t - x)\log\left(\frac{\sqrt{t} + \sqrt{x}}{|\sqrt{t} - \sqrt{x}|}\right),$$

$$g_4(x) = \left(\frac{1+k}{2\pi}\right)\left[(x - 1 - \alpha)\log\left(\frac{(1+\alpha)^{\frac{1}{2}} + \sqrt{x}}{|(1+\alpha)^{\frac{1}{2}} - \sqrt{x}|}\right) - (x - \gamma)\log\left(\frac{\sqrt{x} + \sqrt{\gamma}}{|\sqrt{x} - \sqrt{\gamma}|}\right) + 2\sqrt{x}\left((1+\alpha(^{\frac{1}{2}} - \sqrt{\gamma})\right)\right].$$

The scheme is to be applied for $i > 0$ and $0 \leqslant j \leqslant M_1$, $N_0 \leqslant j \leqslant N_1$ with $S^{(0)}(x)$ 'guessed'; when the solution has converged, values of x lying in the regions $[0, 1]$ and $[1 + \alpha, \gamma]$ may be used in (15) to complete the solution.

Once again, for a given geometry and (subcritical) suction k, the two mass flows M and N must be determined. Two extra mass-flow conditions are therefore required. As in the critical case, over the suction slot the scaled non-dimensional stream function is given by

$$\psi = \frac{M(y - T)}{S - T}.$$

Applying Bernoulli's equation in the same manner as before, we find once again that (11) applies. Finally, since the pressure must recover to p_∞ as $x \to \infty$, it must be the case from (13) that

$$N = S(\infty)$$

The two mass-flow relationships provide a simple strategy for solving the subcritical problem; for a given geometry and a given suction k, initial values $M^{(0)}$ and $N^{(0)}$ of the two mass flows are guessed. The integrodifferential equation is solved in the manner described above, and new values of M and N are generated via

$$M^{(i)} = S^{(i)}(1+\alpha)(1+k)^{\frac{1}{2}}, \qquad N^{(i)} = S^{(i)}(\delta) \quad (i > 0).$$

This simple iterative scheme works efficiently in practice, and invariably leads quickly to a subcricital solution.

It should be noted that for this case, additional relationships may be derived connecting the values of the mass flows and the values of S at various positions. The simplest of these is obtained by multiplying both sides of (13) by $S'(x)$ and integrating with respect to x from 0 to ∞. The left-hand side is identically zero, leaving

$$0=\int_0^\infty S'(x)\,dx-\int_1^{1+\alpha}\frac{M^2}{S^2(x)}S'(x)\,dx$$
$$-(1+k)\int_{1+\alpha}^{\gamma}S'(x)\,dx-\int_\gamma^\infty\frac{N^2}{S^2(x)}S'(x)\,dx.$$

Simplifying this result and using $S(\infty)=N$, $M^2=(1+k)S^2(1+\alpha)$ and $S(0)=0$, gives the condition

$$\frac{N^2}{S(\gamma)}-2N=(1+k)\left[2S(1+\alpha)-S(\gamma)-\frac{S^2(1+\alpha)}{S(1)}\right]. \tag{16}$$

Numerically, it may be confirmed that (16) is satisfied for a converged solution of the subcritical problem, thereby providing an extra check on the solution. It is worth pointing out that, although similar procedures may be carried out for the critical case, because of the square-root singularity in the derivative of S at $x=\gamma$, the resulting formulae involve integrals that cannot be calculated in closed form and are thus of little practical value.

Figure 5 shows streamlines for a typical case. Once again, the values $\alpha=\beta=1$ were used, together with suction $k=1$, the resulting mass flows

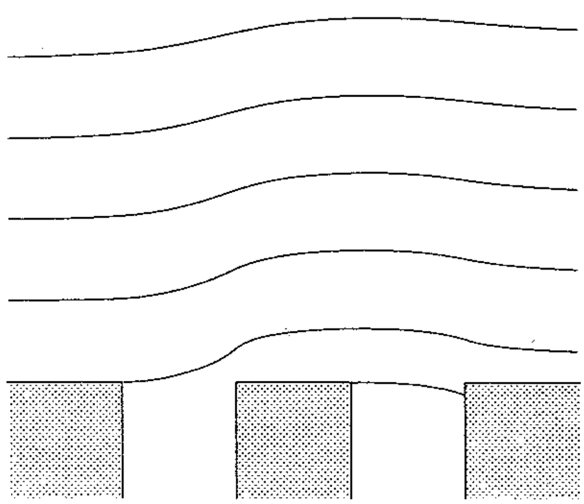

FIG 5 Streamlines for the subcritical suction case with $\alpha=1$, $\beta=1$ and suction $k=1$ (mass flows given by $M=1\ 3927$, $N=0\ 5143$)

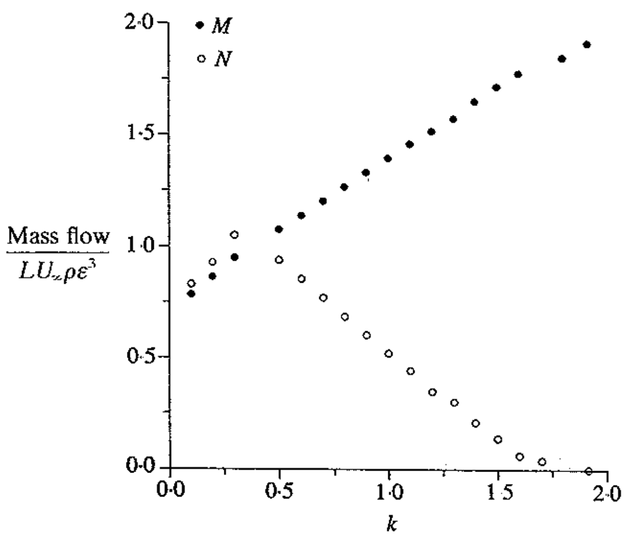

FIG. 6. Relationship between mass flows M, N and the subcritical suction k with $\alpha = 1$, $\beta = 1$

being given by $M = 1{\cdot}3927$, $N = 0{\cdot}5143$. For all subcritical calculations, a value of $\delta = 10$ was used.

Figure 6 shows the relationship between M and N for various subcritical values of k, the geometry being identical to that considered in Fig. 5. We note that, as k decreases to a value of around 0 4070, the mass flows M and N become closer. Although it is possible to continue to perform numerical calculations for even smaller values of k, clearly when N exceeds M, the subcritical model becomes invalid as there is no longer any mass flow into the suction slot. This eventuality is dealt with in the next section.

4. Subsubcritical suction

It was noted in section 3 that, for the case of subcritical suction, when the suction parameter k falls below a critical value for a given geometry, the mass flow N in the layer downstream of the suction slot becomes greater than the mass flow M out of the slot. In this case, the subcritical model as described above is clearly invalid. Physically, however, it is clear what must take place when the suction becomes weak enough for such circumstances to apply: namely, the pressure downstream of the injection slot is reduced to such an extent that the downstream slot, although held at a pressure less than that of the free stream and so nominally a suction slot, acts instead as a second injection slot. Henceforth we refer to this case, in which none of the previously injected flow is reingested into the downstream slot and thus $N > M$ as 'subsubcritical suction'.

Figure 7 shows a schematic diagram of the region over and downstream of the 'suction' slot. Labelling the streamlines and regions as shown in the

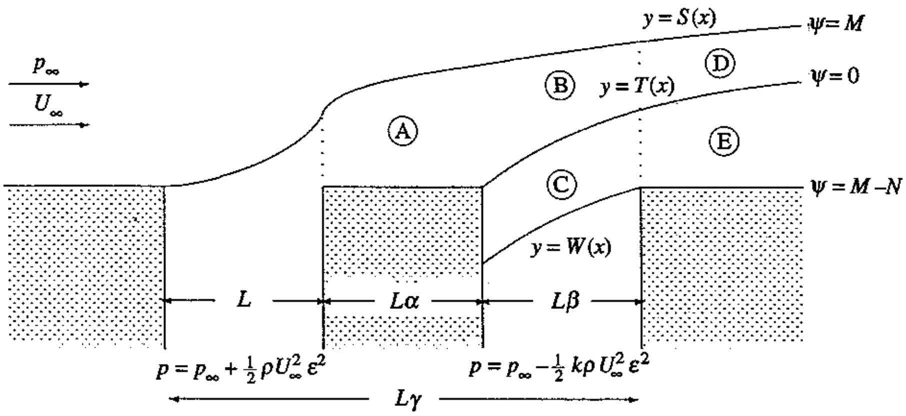

FIG. 7. Schematic diagram of slot injection/suction geometry for the subsubcritical suction case

figure, the relevant scaled dimensionless stream functions are given, to lowest order, by

$$\psi_A = \frac{yM}{S}, \quad \psi_B = \frac{M(y-T)}{S-T}, \quad \psi_C = \frac{(M-N)(y-T)}{W-T},$$

$$\psi_D = \frac{M(y-T)}{S-T}, \quad \psi_E = \frac{y(N-M)}{T} + M - N$$

Matching the Bernoulli constant in regions A, B and D back to the upstream injection slot gives the non-dimensional pressures in these regions as

$$p_A = \frac{p_\infty}{\rho U_\infty^2} + \varepsilon^2\left[\frac{1}{2} - \frac{M^2}{2S^2}\right], \quad p_B = \frac{p_\infty}{\rho U_\infty^2} - \varepsilon^2 \frac{k}{2}, \quad p_D = \frac{p_\infty}{\rho U_\infty^2} + \varepsilon^2\left[\frac{1}{2} - \frac{M^2}{2(S-T)^2}\right],$$

as well as the additional condition

$$M^2 = (1+k)(S-T)^2 \qquad (1+\alpha \leqslant x < \gamma).$$

The relevant singular integrodifferential equation for the subsubcritical case is thus

$$\frac{1}{\pi}\int_0^\infty \frac{S'(t)}{t-x}\,dt = \begin{cases} \frac{1}{2} & (0 \leqslant x < 1), \\ \frac{1}{2} - \frac{1}{2}\frac{M^2}{S^2} & (1 < x < 1+\alpha), \\ -\frac{k}{2} & (1+\alpha \leqslant x < \gamma), \\ \frac{1}{2} - \frac{1}{2}\frac{M^2}{(S-T)^2} & (\gamma < x < \infty), \end{cases} \qquad (17)$$

and another equation is therefore required to close the system.

To obtain this other equation, we note that $T(1+\alpha) = W(\gamma) = 0$. The Bernoulli constant in region C is thus

$$\frac{p_\infty}{\rho U_\infty^2} - \varepsilon^2 \frac{k}{2} + \varepsilon^2 \frac{(M-N)^2}{2(W-T)^2} = \frac{p_\infty}{\rho U_\infty^2} - \varepsilon^2 \frac{k}{2} + \varepsilon^2 \frac{(M-N)^2}{2T^2(\gamma)}.$$

Thus throughout region C we have

$$T(x) - W(x) = \text{constant} = T(\gamma).$$

Applying Bernoulli's equation from region C to E we find that

$$p_E = \frac{p_\infty}{\rho U_\infty^2} + \varepsilon^2 \left[-\frac{k}{2} - \frac{(N-M)^2}{2T^2} + \frac{(N-M)^2}{2T^2(\gamma)} \right],$$

and, since this must be the same as the pressure in region D, it follows that

$$1 + k - \frac{M^2}{(S-T)^2} = (N-M)^2 \left(\frac{1}{T^2(\gamma)} - \frac{1}{T^2} \right)$$

Thus $T(x)$ is determined in terms of $S(x)$ by the quartic equation

$$\begin{aligned} &\big((1+k)T^2(\gamma) - (N-M)^2\big)T^4 + 2S\big((N-M)^2 - T^2(\gamma)(1+k)\big)T^3 \\ &\quad + \big(T^2(\gamma)\big((1+k)S^2 - M^2\big) + (T^2(\gamma) - S^2)(N-M)^2\big)T^2 \\ &\qquad - 2ST^2(\gamma)(N-M)^2 T + S^2 T^2(\gamma)(N-M)^2 = 0 \quad (18) \end{aligned}$$

in the region $\gamma \leqslant x < \infty$, thus closing the system of equations.

Although the equations governing the subcritical and the subsubcritical cases are superficially rather similar, there are major differences between the two flows. For subcritical flow, a vortex sheet of $O(1)$ strength separates the injected flow from the external flow. For subsubcritical flow, this vortex sheet is still present, but an additional vortex sheet (of strength $O(\varepsilon)$) must also be present at $y = T(x)$. To see that this is so, it is easiest to consider the flow variables in regions B and C. Although the static pressures in both regions are equal, the total pressures are not. A vortex sheet must therefore separate the two regions.

Finally, for a given geometry and corresponding subsubcritical value of k, mass-flow conditions are required as usual to determine values for M and N As before, one condition is provided by

$$M = S(1+\alpha)(1+k)^{\frac{1}{2}},$$

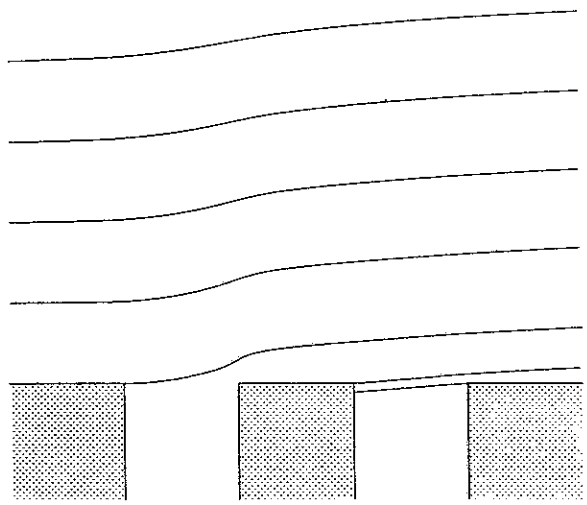

FIG 8. Streamlines for subsubcritical suction case with $\alpha = 1$, $\beta = 1$ and suction $k = 0.1$ (mass flows given by $M = 0.7823$, $N = 0.8293$)

whilst another may be obtained from the fact that, in order for the pressure to recover to p_∞ far downstream of the slots, it is necessary that $S(x) - T(x) \to M$ as $x \to \infty$. From (18), therefore, we have

$$k = (N - M)^2 \left[\frac{1}{T^2(\gamma)} - \frac{1}{T^2(\infty)} \right]$$

so that the condition for N becomes

$$N = M + \left(\frac{kT^2(\gamma)(S(\infty) - M)^2}{(S(\infty) - M)^2 - T^2(\gamma)} \right)^{\frac{1}{2}}$$

Figure 8 shows the streamlines for a subsubcritical case with $\alpha = \beta = 1$ and $k = 0.1$. Once again, a value of $\delta = 10$ was used. The corresponding mass flows were $M = 0.7823$ and $N = 0.8293$, and the flow exhibits typical characteristics of a 'suction' slot behaving as an injection slot. The major difference between this type of injection and the injection from the upstream slot is the absence of a 'lid' effect. This is caused by the fact that for the downstream slot, the effective 'free stream' has only an $O(\varepsilon U_\infty)$ velocity.

It is worth pointing out that, if k is allowed to decrease to values of -1 and below, the analysis presented above fails as the mass-flow formulae become nonsensical. For these values, however, other changes in the assumed orders of magnitude in the flow must also be made and this point will not be pursued further.

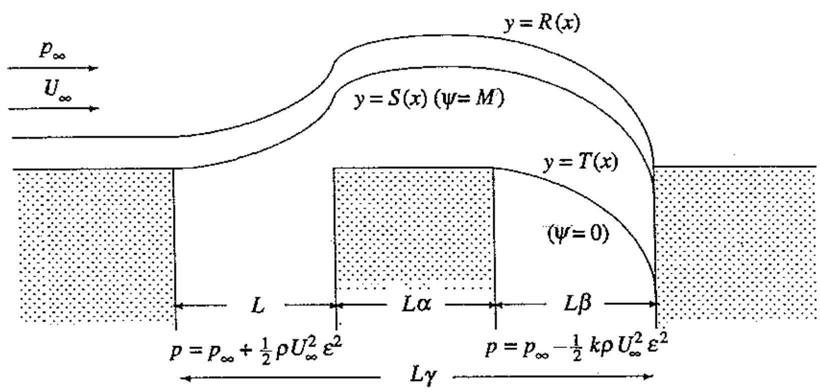

FIG. 9. Schematic diagram of slot injection/suction geometry for the supercritical suction case

5. Supercritical suction

Having investigated flow regimes where the suction parameter k is less than the critical value, we now turn to cases where the suction is stronger. When k exceeds the critical value, some of the free stream flow is ingested by the slot in addition to the injected flow. Figure 9 shows a schematic diagram of the flow, and, by similar reasoning to that used already, the equation satisfied by $S(x)$ is clearly (5) with $S(0) = S'(0) = 0$. In contrast to the critical case, it is no longer true that $S(\gamma) = 0$, whilst $y = R(x)$ (the streamline on which, say, $\psi = Q$) is a 'distinguished streamline' of height $O(\varepsilon^2 L)$ in the flow which determines how much of the free stream is ingested by the suction slot. Bernoulli's equation may be used as before to conclude that $R - S$ is constant, and also that

$$Q = -S(\gamma) \quad \text{and} \quad M = S(1+\alpha)(1+k)^{\frac{1}{2}}. \tag{19}$$

The procedure for determining the flow in the supercritical case is now clear; for a given geometry and supercritical k, we determine the solution where M and $S(1+\alpha)$ are related according to (19). The dimensional mass flow out of the injection slot is given, as usual by $\varepsilon^3 MLU_\infty\rho$, but the mass flow into the suction slot is $-\varepsilon^2 S(\gamma)LU_\infty\rho$. The increase in order of magnitude over the critical suction case arises from the fact that fluid with velocity $O(U_\infty)$ now enters the suction slot.

Figure 10 shows a typical supercritical suction case with $\alpha = \beta = 1$, and $k = 2{\cdot}5$. The associated mass flows were $M = 2{\cdot}2739$ and $Q = 0{\cdot}3917$.

6. Supersupercritical suction

A final case may be considered where k becomes an order of magnitude larger than 1. In this case (which we refer to as supersupercritical suction), the mass flow into the suction slot once again increases by an order of

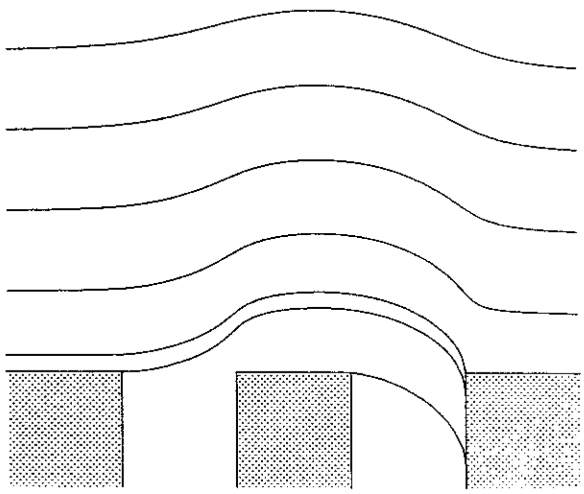

FIG. 10. Streamlines for the supercritical suction case with $\alpha = \beta = 1$, and $k = 2.5$ (non-dimensional scaled mass flows given by $M = 2.2739$, $Q = 0.3917$)

magnitude. We consider specifically the case where $k = K/\varepsilon$ (and $K = O(1)$) so that the suction-slot pressure is given by $p_\infty - \frac{1}{2}K\varepsilon\rho U_\infty^2$. Since, to order ε, the pressure in the injection slot is now p_∞, the injection makes no difference to the flow to leading order and the problem becomes one of pure suction. Since the suction is now, to lowest order, solely from a free stream of velocity $O(U_\infty)$, no vortex sheet is present across $y = S(x)$, and we may expect a layer of thickness $O(L\varepsilon)$ to be ingested by the slot from the free stream. The mass flow in this case is therefore $O(LU_\infty\varepsilon)$. For slot suction at *arbitrary* suction pressures, it has been shown (see (**10**)) that, though the details are somewhat involved, the flow may be determined by hodograph methods. In the supersupercritical suction case, which may be regarded as a special case of this problem, the solution is, however, especially simple to obtain.

Proceeding as in the previously considered cases above, with nomenclature as given in Fig. 9, the pressure match over the suction slot shows that, for $1 + \alpha \leqslant x \leqslant \gamma$, $T(x)$ satisfies

$$\frac{1}{\pi}\int_{1+\alpha}^{\gamma} \frac{T'(t)}{t - x}\,dt = -\frac{K}{2}. \tag{20}$$

Solving (20) subject to the boundary conditions $T(1 + \alpha) = T'(1 + \alpha) = 0$, we find that

$$T(x) = -\frac{K}{2}\left[-\{(x - 1 - \alpha)(\gamma - x)\}^{\frac{1}{2}} + \beta \sin^{-1}\left\{\frac{x - 1 - \alpha}{\beta}\right\}^{\frac{1}{2}}\right].$$

As usual, this streamline reattaches to the downstream wall of the suction slot (at the point $y = -K\pi\beta/4$). The mass flow M may be determined by applying Bernoulli's equation to yield

$$M^2 = (S - T)^2$$

and since, in particular $T(\gamma) = -K\pi\beta/4$ and $S(\gamma) = 0$, the dimensional mass flow in this case is given by

$$M = -\tfrac{1}{4}K\pi\beta L U_\infty \varepsilon.$$

7. Conclusions and discussion

Using what amounts to thin aerofoil theory, the effects of the combined action of injection and suction slots of given positions, dimensions and strengths upon a cross-flow have been analysed. A total of five different regimes have been identified, each involving qualitatively different flow details. Figure 11 shows the relevant mass flows for a case where the suction-slot pressure is increased from a subsubcritical value all the way up to a supersupercritical. For the purposes of presentation, ε was once again

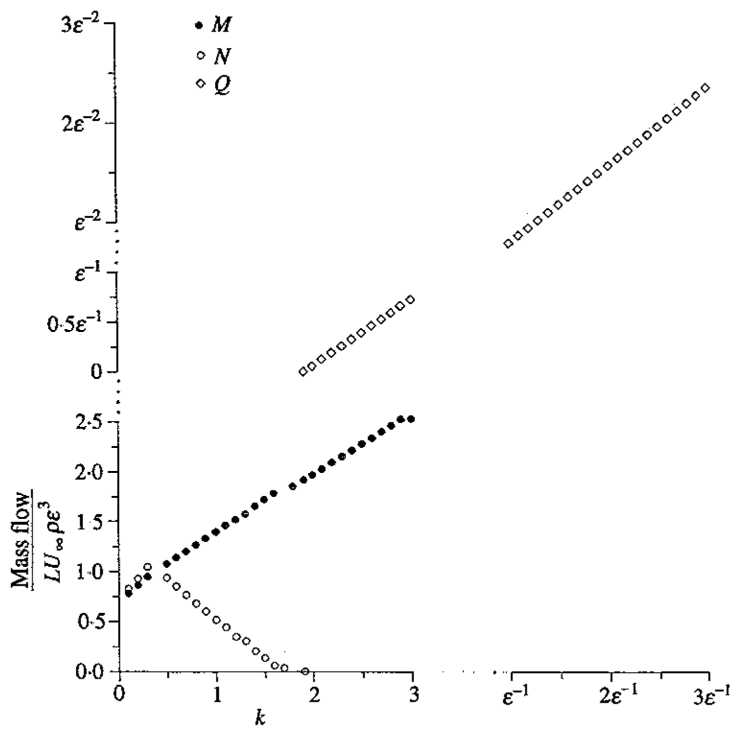

FIG. 11. Relationship between mass flows M, N and Q with $\alpha = \beta = 1$ for values of k ranging from subsubcritical to supersupercritical

taken to be 0·3, and the values $\alpha = \beta = 1$ were used. Convergence of the various schemes was fast in all cases except those where k was slightly less than the critical suction value. In these cases, much smaller relaxations were required. Such behaviour is much as expected, since for such values of k the layer downstream of the suction slot may be expected to be very thin.

In Fig. 11, an implicit assumption was made to the effect that changes in k occur on a time scale of sufficient length to allow the flows to be considered as a succession of quasi-steady problems. Since the slowest characteristic speed involved in the flow is εU_∞, this assumption is justified so long as the time scale exceeds $O(L/(\varepsilon U_\infty))$. For changes in the slot pressures that take place more quickly, a full unsteady problem would have to be considered.

Another interesting observation arising from Fig. 11 concerns the fact that, for subsubcritical values of k, both M and N decrease as k decreases. This is explained by the fact that, for smaller values of k, the flow is effectively partially prevented from escaping from the upstream injection slot. It should also be noted that increasing the pressure at the top of the downstream slot need not necessarily lead to an increase in the pressure at the point from which the fluid that exits the slot emanates.

Finally, it should be recognized that the steady solutions that have been derived may be unstable (see (**11**)). In practice, this may lead to the thickening and roll-up of the vortex sheet separating the injected and cross flows. For suitably large distances downstream, therefore, the theory may be invalid.

REFERENCES

1. S. Hubbard and N. Riley, *Int. J Heat Mass Transfer* **38** (1995) 3209.
2. A D Fitt, J. R. Ockendon and T. V. Jones, *J. Fluid Mech.* **160** (1985) 15.
3. J. D. Cole and J. Aroesty, *Int. J Heat Mass Transfer* **11** (1968) 1167.
4. F. T. Smith and K Stewartson, *J. Fluid Mech.* **58** (1973) 143.
5. M. Van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic Press, New York 1964).
6. L. C. Woods, *The Theory of Subsonic Plane Flow* (University Press, Cambridge 1961).
7. E. Varley and J. D A. Walker, *I M A J. appl. Math.* **43** (1989) 11.
8. F. D. Gakhov, *Boundary Value Problems* (Pergamon, Oxford 1966).
9. N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen 1946).
10. J N. Dewynne, S. D. Howison, J R. Ockendon and E. J. Watson, *J Fluid Mech.* **200** (1989) 265.
11. S. N Brown, H. K Cheng and F T Smith, *ibid.* **193** (1988) 191.