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Group invariant solution for a pre-existing fluid-driven fracture in impermeable rock

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Abstract. The propagation of a two-dimensional fluid-driven fracture in impermeable rock is considered. The fluid flow in the fracture is laminar. By applying lubrication theory a partial differential equation relating the half-width of the fracture to the fluid pressure is derived. To close the model the PKN formulation is adopted in which the fluid pressure is proportional to the half-width of the fracture. By considering a linear combination of the Lie point symmetries of the resulting non-linear diffusion equation the boundary value problem is expressed in a form appropriate for a similarity solution. The boundary value problem is reformulated as two initial value problems which are readily solved numerically. The similarity solution describes a pre-existing fracture since both the total volume and length of the fracture are initially finite and non-zero. Applications in which the rate of fluid injection into the fracture and the pressure at the fracture entry are independent of time are considered.

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1. Introduction

The analysis of a fluid-driven fracture in rock has many applications in science and engineering. Fracture mechanisms may be responsible for the formation of intrusive dikes and sills and for the migration of magma through the lithosphere [1–4]. In oil recovery processes, pumped water is used to enlarge underground fractures [5]. It has also been proposed that ultra-high pressure water could be used to open fissures in rock in mining.

In this paper a similarity solution is derived for a two-dimensional fluid-driven fracture in rock. The two-dimensional model implies that the fracture is infinitely long in the third direction. In applications the breadth is finite and the solution will apply only if the breadth is sufficiently large that end effects can be neglected. The fracture is one-sided and the fluid, which is incompressible, is injected into the fracture at one end. The rock is assumed to be impermeable and therefore leak-off of the fracturing fluid is neglected. It is also assumed that the fluid flow in the fracture is laminar. In some applications the flow will be turbulent [2]. The extension to turbulent flow of the new method of solution that we consider could be investigated. It is also assumed that lubrication theory is applicable.

The pressure of the fluid in the fracture is not determined by lubrication theory. The closure of the equations can be achieved by considering the elasticity of the rock. If the displacement gradients in the rock are small, linear elasticity can be used [6,7]. We will adopt the PKN model in which the fluid pressure is proportional to the half-width of the fracture. The model was first developed by Perkins and Kern [8] and Nordgren [9].

The initial volume and length of the fracture are finite and non-zero. The similarity solution therefore describes the propagation of a fluid-driven pre-existing fracture. The pre-existing nature of a fluid-driven fracture does not seem to have attracted much attention in the literature. The solution may be particularly valuable for processes which depend on the existence of pre-existing fractures, such as the fracturing of rock by ultra-high pressure water. The similarity solution contains an undetermined parameter which can be used to impose on the solution a range of conditions at the fracture entry. The condition that the rate of fluid injection into the fracture is independent of time and the alternative condition that the pressure at the fracture entry is independent of time can both be imposed on the similarity solution.

An outline of the paper is as follows. In Section 2 the equations describing a two-dimensional fluid-driven fracture are formulated using lubrication theory. A differential equation relating the half-width of the fracture to the fluid pressure is obtained. In Section 3 the PKN fracture hypothesis is invoked which yields a nonlinear diffusion equation for the half- width of the fracture. By considering a linear combination of the Lie point symmetries of the nonlinear diffusion equation a group invariant solution is derived which leads to a similarity solution for the propagation of a two-dimensional fluid-driven fracture. The boundary value problem is reformulated as two initial value problems. In Section 4 the general properties of the similarity solution are investigated. Applications of the similarity solution when the rate of fluid injection into the fracture is independent of time and when the pressure at the fracture entry is independent of time are considered. Finally, concluding remarks are made in Section 5.

2. Two-dimensional fluid-driven fracture

We consider a two-dimensional fluid-driven fracture model for impermeable rock. The two-dimensional model was first developed by Khristianovic and Zheltov [10]. A review of hydraulic fracture modelling has been given by Mendelsohn [11]. The nomenclature and coordinate system used are illustrated in Figure 1. Since the rock is impermeable there is no leak-off of the injected fluid. The fracture is one-sided and propagates in the positive x-direction. It is identical in every plane, $y =$

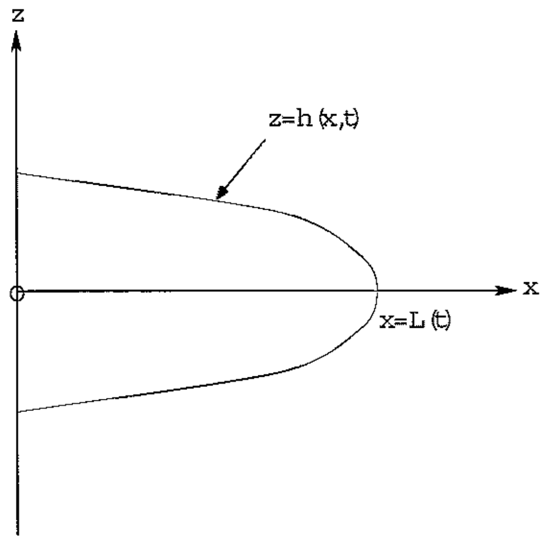

Figure 1. Schematic diagram of the coordinate system and nomenclature used for two-dimensional fracture modelling.

constant, and at time t it has length $L(t)$ and half-width $h(x,t)$. The boundary of the fracture is $z = \pm h(x,t)$ with $0 \leq x \leq L(t)$. The fluid flow in the fracture is independent of y and obeys the Navier–Stokes and conservation of mass equations

$$\mathbf{q}_t + (\mathbf{q} \cdot \nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{q}, \quad \nabla \cdot \mathbf{q} = 0, \tag{2.1}$$

where $\mathbf{q} = (u(x,z,t), 0, w(x,z,t))$ denotes the fluid velocity, the fluid pressure is denoted by $p(x,z,t)$ and it is assumed that body forces may be neglected. To simplify (2.1) for a thin fracture we introduce the dimensionless variables of lubrication theory [12]. Since the length of the fracture is much greater than its width, two length scales are used, L_0, a typical fracture length and H, a typical fracture half-width. Later, L_0 is chosen to be the initial length of the pre-existing fracture. We set $x = L_0\bar{x}$, $h = H\bar{h}$, $z = H\bar{z}$, $u = U\bar{u}$, $\bar{w} = \varepsilon U \bar{w}$, $t = (L_0/U)\bar{t}$ and $p = (\mu U / L_0 \varepsilon^2)\bar{p}$ where $\varepsilon = H/L_0$ and U is a typical fluid speed in the fracture in the x-direction. With these scalings, (2.1) become, dropping the overhead bars for simplicity,

$$\varepsilon^2 \mathrm{Re}(u_t + uu_x + wu_z) = -p_x + \varepsilon^2 u_{xx} + u_{zz}, \tag{2.2}$$

$$\varepsilon^4 \mathrm{Re}(w_t + uw_x + ww_z) = -p_z + \varepsilon^4 w_{xx} + \varepsilon^2 w_{zz}, \tag{2.3}$$

$$u_x + w_z = 0, \tag{2.4}$$

where $\mathrm{Re} = \frac{UL_0}{\nu}$ is the Reynolds number. We make the thin fluid film approximation of lubrication theory [12],

$$\varepsilon = \frac{H}{L_0} \ll 1, \qquad \varepsilon^2 \mathrm{Re} = \frac{UH^2}{\nu L_0} \ll 1. \tag{2.5}$$

The leading order terms in (2.2) to (2.4) are those of standard two-dimensional lubrication theory:

$$p_x = u_{zz}, \quad p_z = 0, \quad u_x + w_z = 0. \tag{2.6}$$

The boundary conditions at $z = \pm h(x,t)$ are the no-slip condition for a viscous fluid at a solid boundary and the no leak-off condition, namely

$$u(x, \pm h(x,t), t) = 0, \quad w(x, \pm h(x,t), t) = \left. \frac{Dz}{Dt} \right|_{z=\pm h} = \pm \frac{\partial h}{\partial t}, \tag{2.7}$$

where $\frac{D}{Dt}$ denotes the material time derivative.

Integration of the first two equations in (2.6) subject to the boundary conditions (2.7) yields $p = p(x,t)$ and

$$u(x,z,t) = -\frac{1}{2}\left(h^2(x,t) - z^2\right) \frac{\partial p}{\partial x}(x,t) \tag{2.8}$$

Integrating the last equation in (2.6) across the fracture, using the formula for differentiation under the integral sign [13] and the boundary conditions (2.7) gives the conservation equation

$$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\int_{-h(x,t)}^{h(x,t)} u(x,z,t)dz = 0. \tag{2.9}$$

Finally, substituting (2.8) into (2.9) yields

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3 \frac{\partial p}{\partial x}\right). \tag{2.10}$$

Consider now the remaining conditions. At the fracture tip,

$$h(L(t), t) = 0 \tag{2.11}$$

Also, let $V(t)$ denote the total volume of the fracture per unit length in the y-direction:

$$V(t) = 2\int_0^{L(t)} h(x,t)dx. \tag{2.12}$$

Since the fluid is incompressible, the time rate of increase of the total volume of the fracture per unit length in the y-direction is equal to the rate of flow of fluid into the fracture per unit length in the y-direction at $x = 0$:

$$\frac{dV}{dt} = 2\int_0^{h(0,t)} u(0,z,t)dz. \tag{2.13}$$

Using (2 8) for $u(0,z,t)$, equation (2.13) becomes

$$\frac{dV}{dt} = -\frac{2}{3}h^3(0,t)\frac{\partial p}{\partial x}(0,t) \tag{2.14}$$

Equation (2 14) gives the time rate of fluid injection into the fracture per unit length in the y-direction A further condition that may be specified is the initial fracture shape:

$$h(x,0) = h_0(x) \tag{2 15}$$

In general it will not be possible for a similarity solution to satisfy (2.15) for arbitrary $h_0(x)$.

The pressure $p(x,t)$ in (2.10) was not determined by lubrication theory It is determined from the elasticity of the rock. Many alternative models have been proposed including linear elasticity if the displacement gradients in the rock are small. We will adopt the PKN model [8,9] which relates $p(x,t)$ to $h(x,t)$ and yields a closed system of equations

3. Group invariant solution for the PKN model

Various arguments have been advanced to justify PKN theory [8,9], but essentially proponents of PKN theory assume that

$$p(x,t) = \Lambda h(x,t), \tag{3.1}$$

where Λ is a constant that may be calculated from the material properties of the rock. Equation (3 1) is applicable in lubrication theory because p is independent of z.

Using (3.1) as a fracture constitutive law for the two-dimensional model and redefining time as $t' = \Lambda t$ and suppressing the dash, (2.10) now becomes the nonlinear diffusion equation

$$\frac{\partial h}{\partial t} = \frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right) \tag{3.2}$$

for the fracture half-width $h(x,t)$. Equation (3.2) must be solved subject to the boundary condition (2.11) where the characteristic length is chosen so that $L(0) = 1$ and subject also to the condition (2.14) where $V(t)$ is given by (2.12). We will derive a similarity solution by considering the group invariant solution of (3.2) Although it will not be possible to satisfy the initial condition (2.15) for arbitrary $h_0(x)$, crucially we will impose the condition $L(0) = 1$ so that similarity solutions for "pre-existing fractures" may be simulated.

The Lie point symmetry generators of (3 2) may be derived by standard methods [14]:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial t} - \frac{1}{3}h\frac{\partial}{\partial h},$$
$$X_3 = x\frac{\partial}{\partial x} + \frac{2}{3}h\frac{\partial}{\partial h}, \qquad X_4 = \frac{\partial}{\partial x} \tag{3 3}$$

We consider a linear combination of the Lie point symmetry generators,

$$X = c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4, \tag{3.4}$$

where c_1 to c_4 are unknown constants. Only ratios of the constants will be determined because a constant multiple of X is also a Lie point symmetry generator. Now, $h = \Phi(x,t)$ is a group invariant solution of (3.2) provided

$$X\left(h - \Phi(x,t)\right)\Big|_{h=\Phi} = 0, \tag{3.5}$$

that is, provided

$$(c_1 + c_2 t)\frac{\partial \Phi}{\partial t} + (c_4 + c_3 x)\frac{\partial \Phi}{\partial x} = \frac{1}{3}(2c_3 - c_2)\Phi. \tag{3.6}$$

The general solution of (3.6) yields, since $\Phi = h$,

$$h(x,t) = (c_1 + c_2 t)^{\frac{2}{3}\frac{c_3}{c_2} - \frac{1}{3}} F(\xi), \tag{3.7}$$

where $F(\xi)$ is an arbitrary function of ξ and

$$\xi = \frac{c_4 + c_3 x}{(c_1 + c_2 t)^{c_3/c_2}}. \tag{3.8}$$

If (3.7) is substituted into (3.2) then the following ordinary differential equation for $F(\xi)$ is derived:

$$c_3 \frac{d}{d\xi}\left(F^3 \frac{dF}{d\xi}\right) + 3\frac{d}{d\xi}(\xi F) + \left(\frac{c_2}{c_3} - 5\right) F = 0 \tag{3.9}$$

We choose $c_4 = 0$ so that $\xi = 0$ when $x = 0$. The boundary condition (2.11) becomes

$$F\left(\frac{c_3 L(t)}{(c_1 + c_2 t)^{c_3/c_2}}\right) = 0 \tag{3.10}$$

If (3.7) is substituted into (2.12) then we obtain

$$V(t) = \frac{2}{c_3}(c_1 + c_2 t)^{\frac{5}{3}\frac{c_3}{c_2} - \frac{1}{3}} \int_0^{\frac{c_3 L(t)}{(c_1 + c_2 t)^{c_3/c_2}}} F(\xi)\, d\xi. \tag{3.11}$$

Condition (2.14) now becomes, with the aid of the formula for differentiation under the integral sign [13] and the boundary condition (3.10),

$$c_3 F^3(0)\frac{dF}{d\xi}(0) = \left(\frac{c_2}{c_3} - 5\right) \int_0^{\frac{c_3 L(t)}{(c_1 + c_2 t)^{c_3/c_2}}} F(\xi)\, d\xi. \tag{3.12}$$

The upper limit in the integral in (3.12) must be constant and imposing $L(0) = 1$ we find that

$$L(t) = \left(1 + \frac{c_2}{c_1} t\right)^{c_3/c_2} \tag{3.13}$$

and (3.12) reduces to

$$c_3 F^3(0)\frac{dF}{d\xi}(0) = \left(\frac{c_2}{c_3} - 5\right)\int_0^{c_3 c_1^{-c_3/c_2}} F(\xi)d\xi \tag{3.14}$$

Equation (3 11) can be written as

$$V(t) = V_0\left(1 + \frac{c_2}{c_3}t\right)^{\frac{5}{3}\left(\frac{c_3}{c_2} - \frac{1}{5}\right)}, \tag{3 15}$$

where

$$V_0 = \frac{2}{c_3}c_1^{\frac{5}{3}\left(\frac{c_3}{c_2} - \frac{1}{5}\right)}\int_0^{c_3 c_1^{-c_3/c_2}} F(\xi)d\xi \tag{3 16}$$

and V_0 is the total volume of the fracture per unit length in the y-direction at $t = 0$

In order to further simplify the equations we make the change of variables

$$\xi = c_3 c_1^{-c_3/c_2}u, \qquad u = \frac{x}{L(t)}, \qquad F(\xi) = c_3^{\frac{1}{3}} c_1^{-\frac{2}{3}\frac{c_3}{c_2}} G(u), \tag{3.17}$$

where $0 \leq u \leq 1$. The problem can be reformulated as follows·

$$\frac{d}{du}\left(G^3\frac{dG}{du}\right) + 3\frac{d}{du}(uG) + \left(\frac{c_2}{c_3} - 5\right)G = 0, \tag{3.18}$$

$$G(1) = 0, \tag{3 19}$$

$$G^3(0)\frac{dG}{du}(0) = \left(\frac{c_2}{c_3} - 5\right)\int_0^1 G(u)du, \tag{3.20}$$

$$V_0 = 2\left(\frac{c_3}{c_1}\right)^{\frac{1}{3}}\int_0^1 G(u)du, \tag{3 21}$$

$$\frac{c_2}{c_1} = \frac{c_2}{c_3}\frac{c_3}{c_1}, \tag{3 22}$$

$$V(t) = V_0\left(1 + \frac{c_2}{c_1}t\right)^{\frac{5}{3}\left(\frac{c_3}{c_2} - \frac{1}{5}\right)}, \tag{3 23}$$

$$L(t) = \left(1 + \frac{c_2}{c_1}t\right)^{c_3/c_2}, \tag{3 24}$$

$$h(x,t) = \left(\frac{c_3}{c_1}\right)^{\frac{1}{3}}\left(1 + \frac{c_2}{c_1}t\right)^{\frac{2}{3}\left(\frac{c_3}{c_2} - \frac{1}{2}\right)} G(u) \tag{3 25}$$

The rate of fluid injection into the fracture per unit length in the y-direction is from (3 23),

$$\frac{dV}{dt} = \frac{5}{3}\frac{c_2}{c_1}\left(\frac{c_3}{c_2} - \frac{1}{5}\right)V_0\left(1 + \frac{c_2}{c_1}t\right)^{\frac{5}{3}\left(\frac{c_3}{c_2} - \frac{4}{5}\right)} \tag{3 26}$$

From (3.1) and (3 25), the fluid pressure in the fracture is

$$p(x,t) = \Lambda \left(\frac{c_3}{c_1}\right)^{\frac{1}{3}} \left(1 + \frac{c_2}{c_1}t\right)^{\frac{2}{3}\left(\frac{c_3}{c_2} - \frac{1}{2}\right)} G(u) \qquad (3.27)$$

It can be verified that for all values of c_3/c_2 the asymptotic solution as $u \to 1$ of (3.18) that satisfies (3.19) is

$$G(u) \sim 3^{2/3}(1-u)^{1/3}, \quad \text{as} \quad u \to 1. \qquad (3\ 28)$$

The general procedure for solution is as follows The ratio c_3/c_2 is first determined from further information For instance, if the rate of fluid injection into the fracture is specified then c_3/c_2 is determined from (3 26). If the pressure at the fracture entry is specified then c_3/c_2 is determined from (3.27) The ordinary differential equation (3.18) is then solved for $G(u)$ subject to the boundary conditions (3 19) and (3.20) The ratio c_3/c_1 is calculated from (3.21) in terms of V_0 which is given and the ratio c_2/c_1 is then obtained from (3.22) Finally, $V(t)$, $L(t)$ and $h(x,t)$ are calculated from (3.23), (3.24) and (3 25).

The range of values of c_3/c_2 which we will consider is $0.2 \le c_3/c_2 \le 1$. From (3 26), for $c_3/c_2 < 0.2$, the total volume of the fracture per unit breadth, $V(t)$, is a decreasing function of time which does not describe hydraulic fracture. For $c_3/c_2 = 0.2$, $V(t)$ remains constant. For $c_3/c_2 = 1\ 0$, the speed of fracture propagation, dL/dt, is constant. For $c_3/c_1 > 1$, dL/dt is an increasing function of time which we do not expect to apply in hydraulic fracture. Contained in this range are two important special cases. When the rate of fluid injection into the fracture is independent of time it follows from (3.26) that $c_3/c_2 = 0\ 8$. Also, from (3.27), the pressure at the fracture entry is

$$p(0,t) = \Lambda \left(\frac{c_3}{c_1}\right)^{\frac{1}{3}} \left(1 + \frac{c_2}{c_1}t\right)^{\frac{2}{3}\left(\frac{c_3}{c_2} - \frac{1}{2}\right)} G(0) \qquad (3\ 29)$$

If the pressure at the fracture entry is independent of time then $c_3/c_2 = 0.5$. Exact analytical solutions can be derived for the limiting cases $c_3/c_2 = 0.2$ and $c_3/c_2 = 1$.

Consider first $c_3/c_2 = 0.2$ for which $V(t)$ takes the constant value V_0. Equation (3.18) can be integrated immediately and the boundary condition (3 19) at $u = 1$ can be imposed since the asymptotic behaviour of $G(u)$ as $u \to 1$ is known from (3 28). The solution is

$$G(u) = \left(\frac{9}{2}\right)^{1/3} (1-u^2)^{1/3},$$

$$\frac{c_3}{c_1} = \frac{1}{36}\left(\frac{V_0}{I}\right)^3, \qquad \frac{c_2}{c_1} = \frac{5}{36}\left(\frac{V_0}{I}\right)^3, \qquad I = \int_0^1 (1-u^2)^{1/3} du = 0.8413,$$

$$V(t) = V_0, \quad L(t) = \left[1 + \frac{5}{36}\left(\frac{V_0}{I}\right)^3 t\right]^{1/5}, \qquad h(x,t) = \frac{V_0}{2IL(t)}\left[1 - \frac{x^2}{L^2(t)}\right]^{1/3}. \qquad (3\ 30)$$

Consider next $c_3/c_2 = 1$ for which dL/dt takes the constant value c_2/c_1. The solution is

$$G(u) = 3^{2/3}(1-u)^{1/3},$$
$$\frac{c_3}{c_1} = \frac{c_2}{c_1} = \frac{8}{243}V_0^3, \qquad V(t) = V_0\left[1+\frac{8}{243}V_0^3 t\right]^{4/3}, \tag{3.31}$$
$$L(t) = 1 + \frac{8}{243}V_0^3 t, \qquad h(x,t) = \frac{2}{3}V_0 L(t)^{1/3}\left[1-\frac{x}{L(t)}\right]^{1/3}.$$

The differential equation (3 18) has only one Lie point symmetry generator

$$X = 3u\frac{\partial}{\partial u} + 2G\frac{\partial}{\partial G}. \tag{3 32}$$

In general it therefore cannot be integrated completely to give an analytical solution. It is solved numerically for $0.2 < c_3/c_2 < 1$ The generator (3 32), however, is a scaling symmetry and it can be used to transform the boundary value problem, (3.18)) to (3.20), into two initial value problems which are easier to solve. This was first done by Blasius for the boundary value problem of steady two-dimensional flow along a flat plate [15]. The technique has been extended in several ways [16-20]. To rewrite the boundary value problem, (3 18) to (3.20), as a pair of initial value problems we first observe, using Lie's equations [14], that (3.32) generates the transformation

$$\bar{u} = ue^{3a}, \quad \bar{G} = Ge^{2a}, \tag{3.33}$$

where a is the group parameter Hence G satisfies the scaling transformation

$$G(u) = \lambda^{-2/3}\bar{G}(\lambda u) \tag{3.34}$$

where $\lambda = e^{3a}$ The transformation (3 34) leaves the form of the differential equation (3.18) invariant. Also

$$G(0) = \lambda^{-2/3}\bar{G}(0). \tag{3.35}$$

We choose $\bar{G}(0) = 1$ and therefore $G(0) = \lambda^{-2/3}$. The parameter λ is obtained from the boundary condition (3.19) which becomes $\bar{G}(\lambda) = 0$.

The boundary value problem, (3.18) to (3.20), can therefore be transformed into the following two initial value problems:

Problem 1.

$$\frac{d}{d\bar{u}}\left(\bar{G}^3\frac{d\bar{G}}{d\bar{u}}\right) + 3\frac{d}{d\bar{u}}(\bar{u}\bar{G}) + \left(\frac{c_2}{c_3} - 5\right)\bar{G} = 0, \tag{3 36}$$

$$\bar{G}(0) = 1, \qquad \frac{d\bar{G}}{d\bar{u}}(0) = \left(\frac{c_2}{c_3} - 5\right)\int_0^\lambda \bar{G}(\bar{u})d\bar{u}, \tag{3 37}$$

where $0 \le \bar{u} \le \lambda$ and λ is defined by

$$\bar{G}(\lambda) = 0 \tag{3.38}$$

Problem 2.

$$\frac{d}{du}\left(G^3\frac{dG}{du}\right)+3\frac{d}{du}(uG)+\left(\frac{c_2}{c_3}-5\right)G=0, \tag{3 39}$$

$$G(0)=\lambda^{-2/3}, \qquad \frac{dG}{du}(0)=\lambda^2\left(\frac{c_2}{c_3}-5\right)\int_0^1 G(u)du, \tag{3 40}$$

where $0 \leq u \leq 1$.

The parameter λ is obtained from Problem 1 and $G(u)$ is calculated from Problem 2 which is then substituted into (3.21) to (3 25) to complete the solution. For the limiting case $c_3/c_2 = 0.2$,

$$\bar{G}(\bar{u})=\left(\frac{9}{2}\right)^{1/3}\left(\frac{2}{9}-\bar{u}^2\right)^{1/3}, \qquad \lambda=\frac{\sqrt{2}}{3} \tag{3 41}$$

and $G(u)$ and the remainder of the solution is given by (3.30). For the limiting case $c_3/c_2 = 1$,

$$\bar{G}(\bar{u})=3^{1/3}\left(\frac{1}{3}-\bar{u}\right)^{1/3}, \qquad \lambda=\frac{1}{3} \tag{3.42}$$

and $G(u)$ and the remainder of the solution is given by (3.31).

4. Results and applications

The general properties of the solution will first be investigated. Two special cases will then be considered which are illustrated by three applications that cover a wide range of values of the parameters.

Problems 1 and 2 are solved numerically for $0.2 \leq c_3/c_2 \leq 1$ using standard initial value solvers in MAPLE 9. The calculations are readily performed and may be carried out in a few seconds of CPU time For $c_3/c_2 = 0.2$ and 1, the numerical results were found to agree with the analytical results given by (3.30) and (3 31).

The results (3.23) to (3.25) for $V(t)$, $L(t)$ and $h(x,t)$ depend on c_3/c_2 and V_0 and are expressed in terms of time t which is made dimensionless by division by $L_0/\Lambda U$ The dimensionless parameter V_0 is the initial volume of the fracture per unit breadth in the y-direction divided by HL_0 where H is a typical fracture half-width. We choose $V_0 = 1$ throughout this section.

The numerical results for $0.2 \leq c_3/c_2 \leq 1$ and $V_0 = 1$ are displayed in Table 1 and the fracture length $L(t)$ is plotted against the time t in Figure 2. For $c_3/c_2 = 0.2$ the volume V of the fracture is constant, for $c_3/c_2 = 0.5$ the pressure at the fracture entrance, $p(0,t)$, is constant, for $c_3/c_2 = 0.8$ the rate of fluid injection into the fracture, dV/dt, is constant and for $c_3/c_2 = 1$ the speed of propagation of the fracture, dL/dt, is constant

From Table 1 we see that $dL(0)/dt$ decreases steadily as c_3/c_2 increases from 0.2 to 1.0. The fracture length $L(t)$ is bounded by the analytical results, (3.30)

$\frac{c_3}{c_2}$	$\frac{c_2}{c_1}$	λ	$V(t)$	$L(t)$	$\frac{dL}{dt}(0) = \frac{c_3}{c_1}$
0.2	0.2332	0.4714	1	$(1+0.233t)^{1/5}$	0.0466
0.3	0.1317	0.3942	$(1+0.132t)^{1/6}$	$(1+0.132t)^{3/10}$	0.0395
0.4	0.0920	0.3684	$(1+0.092t)^{1/3}$	$(1+0.092t)^{2/5}$	0.0368
0.5	0.0708	0.3554	$(1+0.071t)^{1/2}$	$(1+0.071t)^{1/2}$	0.0354
0.6	0.0575	0.3475	$(1+0.058t)^{2/3}$	$(1+0.058t)^{3/5}$	0.0345
0.7	0.0485	0.3422	$(1+0.048t)^{5/6}$	$(1+0.048t)^{7/10}$	0.0339
0.8	0.0419	0.3384	$1+0.042t$	$(1+0.042t)^{4/5}$	0.0335
0.9	0.0369	0.3356	$(1+0.037t)^{7/6}$	$(1+0.037t)^{9/10}$	0.0332
1.0	0.0329	0.3333	$(1+0.033t)^{4/3}$	$1+0.033t$	0.0329

Table 1 Parameter values for the fracture lengths $L(t)$ with $V_0 = 1$. The characteristic length is the initial length of the fracture L_0 and the characteristic time is $L_0/U\Lambda$

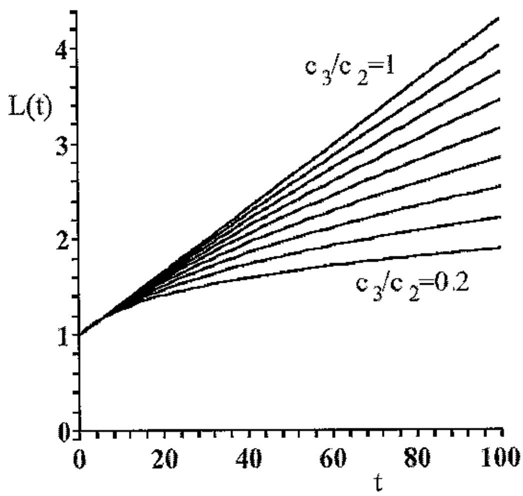

Figure 2. Fracture length $L(t)$ plotted against time t with $V_0 = 1$ and $c_3/c_2 = 0.2, 0.3, \ldots, 1.0$ The characteristic length is the initial length of the fracture L_0 and the characteristic time is $L_0/U\Lambda$.

for $c_3/c_2 = 0.2$ and (3 31) for $c_3/c_2 = 1.0$. For $t \lesssim 5$ and for $0.2 \le c_3/c_2 \le 1$,

$$1 + \frac{8}{243} V_0^3 t \le L(t) \le \left[1 + \frac{5}{36}\left(\frac{V_0}{I}\right)^3 t\right]^{1/5} \tag{4.1}$$

while for $t \gtrsim 5$ the opposite is the case and

$$\left[1 + \frac{5}{36}\left(\frac{V_0}{I}\right)^3 t\right]^{1/5} \le L(t) \le 1 + \frac{8}{243} V_0^3 t. \tag{4.2}$$

By scaled time $t = 100$, the pre-existing fracture with length L_0 has propagated

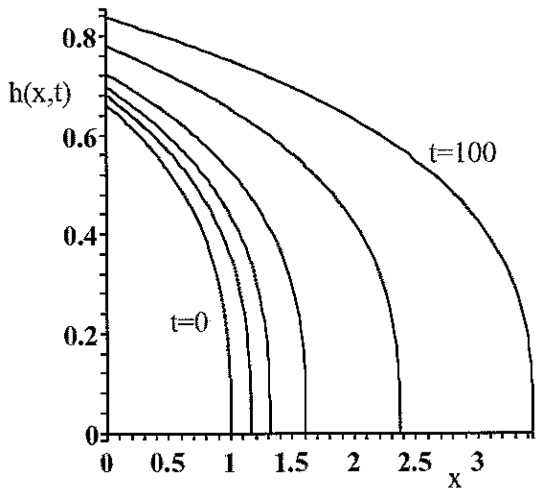

Figure 3 Fracture half-width $h(x,t)$ plotted against x for values $V_0 = 1$, $c_3/c_2 = 0.7$ and $t = 0$, 5, 10, 20, 50 and 100 The characteristic time is $L_0/U\Lambda$

to length $1.89L_0$, $2.85L_0$, $3.74L_0$ and $4.3L_0$ for $c_3/c_2 = 0.2$, 0.5, 0.8 and 1.0 respectively

In Figure 3, $h(x,t)$ is plotted against x for $V_0 = 1$ and for a range of values of time with $c_3/c_2 = 0.7$ The spatial gradient of $h(x,t)$ is negative infinity at the fracture tip, $x = L(t)$. This also applies to the analytical solutions (3.30) and (3.31) for $c_3/c_2 = 0.2$ and 1.0. The assumptions (2.5) of lubrication theory consequently break down at the fracture tip.

Consider now Λ which occurs in the characteristic time $L_0/\Lambda U$. Expressed in dimensional variables, (3 1) is

$$p(x,t) = \Lambda_D h(x,t) \tag{4.3}$$

where Λ_D has dimensions of Pa m^{-1}. We will take [8, 21]

$$\Lambda_D = \frac{E}{(1-\sigma^2)B}, \tag{4.4}$$

where E and σ are the Young's modulus and Poisson ratio of the rock and B is the breadth in the y-direction of the fracture. Since the characteristic pressure of lubrication theory is

$$P = \frac{\mu U L_0}{H^2}, \tag{4.5}$$

the dimensionless parameter Λ is related to Λ_D by

$$\Lambda = \frac{H^3}{\mu U L_0}\Lambda_D. \tag{4.6}$$

Substituting (4.4) into (4.6) yields

$$\Lambda = \frac{EH^3}{(1-\sigma^2)\mu U L_0 B}. \tag{4.7}$$

In order to obtain numerical results we transform back to dimensional variables. The characteristic length is L_0 and the characteristic time is $L_0/\Lambda H$. Equation (3.24) expressed in dimensional variables is

$$L(t) = L_0 \left[1 + \frac{c_2}{c_1}\frac{\Lambda U}{L_0} t\right]^{c_3/c_2}. \tag{4.8}$$

The ratio c_2/c_1 depends on V_0 and on c_3/c_2. It is given in Table 1 for $V_0 = 1$ and $0.2 \le c_3/c_2 \le 1$. Using (4.7), (4.8) becomes

$$L(t) = L_0 \left[1 + \frac{c_2}{c_1}\frac{EH^3}{(1-\sigma^2)\mu B L_0^2} t\right]^{c_3/c_2}. \tag{4.9}$$

Lubrication theory will apply throughout the propagation if the conditions (2.5) are satisfied. The Reynolds number for flow at the entry to the fracture is [1, 2]

$$\mathrm{Re}^* = \frac{2HU}{\nu} \tag{4.10}$$

The onset of turbulence is usually taken to be at $\mathrm{Re}^* \simeq 10^3$ [1, 2]. We now consider two important special cases

4.1. Constant rate of fluid injection

When the rate of fluid injection into the fracture is independent of time, $c_3/c_2 = 0.8$. For $c_3/c_2 = 0.8$ and $V_0 = 1$, $c_2/c_1 = 0.042$ and (4.9) becomes

$$L(t) = L_0 \left[1 + 0.042\frac{EH^3}{(1-\sigma^2)\mu B L_0^2} t\right]^{4/5}. \tag{4.11}$$

The speed of propagation of the fracture when $c_3/c_2 = 0.8$ and $V_0 = 1$ is

$$\frac{dL}{dt} = 0.034\frac{EH^3}{(1-\sigma^2)\mu B L_0}\left[1 + 0.042\frac{EH^3}{(1-\sigma^2)\mu B L_0^2} t\right]^{-1/5}. \tag{4.12}$$

The rate of fluid injection into the fracture is

$$A = \frac{dV}{dt}\,\mathrm{m}^2\mathrm{s}^{-1} \tag{4.13}$$

and is a given constant. The characteristic fluid velocity in the x-direction is

$$U = \frac{A}{2H} \tag{4.14}$$

Consider now two applications with A constant.

(i) Formation of dikes and sills

Consider the formation of a dike or sill in the Earth's crust by the injection of magma into a pre-existing fracture which then starts to propagate The magma is supplied at a constant rate through a conduit. Parameter values given by Spence and Turcotte [1] are used and are given in Table 2. These authors did not specify the breadth, B, of the fracture. A reasonable value is $B = 300$ m. The duration of the fluid injection after the fracture starts to propagate is 10^3 s (16 6 min). The results for a pre-existing fracture of length 100 m are presented in Table 3 for $V_0 = 1$ After 10^3 s this fracture has propagated to 2.2 km. The initial speed of propagation of the fracture is 3.7 ms^{-1} and this decreases to 1.7 ms^{-1} after 10^3 s. The characteristic fluid velocity is 2 ms^{-1}. From (4.5), the fluid pressure at the start of propagation is about 0.3 MPa (1 MPa = 10^6 Pa) which increases to about 7 MPa as the length of the fracture increases to 2 2 km. The conditions for lubrication theory and laminar flow are satisfied.

	A m^{2}s^{-1}	P Pa	ρ kg m^{-3}	μ Pa s	ν m^{2}s^{-1}	E Pa	σ	H m	B m	L_0 m	Δt s	$\frac{c_3}{c_2}$
Formation of dikes and skills	1.0		2 5$\times10^{3}$	10^{2}	4×10^{-2}	2×10^{10}	0 25	0 25	300	100	10^{3}	0 8
Enlarge-ment of fractures for oil extraction	5×10^{-3}		10^{3}	9 6$\times10^{-2}$	9 6$\times10^{-5}$	2 6$\times10^{10}$	0 2	2×10^{-3}	15	30	3×10^{3}	0.8
Fracturing rock with ultra-high pressure water (250 MPa)		2.5$\times10^{8}$	10^{3}	8 7$\times10^{-5}$	8 7$\times10^{-8}$	7×10^{10}	0.25	5×10^{-6}	10^{-5}	0.5	5×10^{-4}	0.8
Fracturing rock with ultra-high pressure water (1000 MPa)		10^{9}	10^{3}	8 7$\times10^{-5}$	8 7$\times10^{-8}$	7×10^{10}	0 25	5×10^{-6}	10^{-5}	0.5	2×10^{-3}	0.5

Table 2 Parameter values for fluid-driven fractures

The results are the same order of magnitude as obtained by Spence and Turcotte [1] for a two-sided fracture expanding from a point source. After 10^3 s, the length of each side was 2 km, the propagation speed of the fracture was 1 ms^{-1} and the fluid pressure was 6.3 MPa.

(ii) Enlargement of fractures to enhance oil recovery

The enlargement of fractures by pumping water with additives in order to

	$L(\Delta t)$ m	$\frac{dL}{dt}(0)$ ms^{-1}	$\frac{dL}{dt}(\Delta t)$ ms^{-1}	$\varepsilon = \frac{H}{L_0}$	$U = \frac{A}{2H}$ ms^{-1}	$U = \frac{PH^2}{\mu L_0}$ ms^{-1}	$\varepsilon^2\mathrm{Re} = \frac{UH^2}{\nu L_0}$	$\mathrm{Re}^* = \frac{2HU}{\nu}$
Formation of dikes and skills	2180	3.7	1.7	2.2×10^{-4}	2.0		2.7×10^{-3}	25
Enlargement of fractures for oil extraction	355	0.17	0.09	2.9×10^{-5}	1.25		1.7×10^{-3}	52
Fracturing rock with ultra-high pressure water (250 MPa)	0.79	760	480	10^{-5}		140	8×10^{-2}	1.65×10^{4}
Fracturing rock with ultra-high pressure water (1000 MPa)	1.33	760	286	10^{-5}		575	0.33	6.6×10^{4}

Table 3. Results derived from the parameter values presented in Table 2 for $V_0 = 1$

enhance oil recovery is a well established technique. We will assume that the water is pumped at a constant rate. An estimate of the values of the parameters is given in Table 2 [21]. After the fracture starts to propagate the water is injected for a further 3×10^3 s (50 min). In Table 3 the results when $V_0 = 1$ are presented for a pre-existing fracture of length 30 m. The fracture propagates to 355 m after 3×10^3 s with an initial propagation speed of 0.17 ms^{-1} and a final speed of 0.09 ms^{-1}. The characteristic fluid velocity is 1.25 ms^{-1}. The fluid pressure at the start of the propagation is, from (4.5), about 0.9 MPa and this increases to about 10 MPa when the length of the fracture is 355 m. The assumptions of lubrication theory and the conditions for laminar flow are satisfied. These results are within the expected range of values for hydraulic fracturing.

4.2. Constant pressure at the fracture entry

When the pressure at the fracture entry is independent of time, $c_3/c_2 = 0.5$. For $c_3/c_2 = 0.5$ and $V_0 = 1$, $c_2/c_1 = 0.071$ and (4.9) becomes

$$L(t) = L_0 \left[1 + 0.071 \frac{EH^3}{(1-\sigma^2)\mu B L_0^2} t \right]^{1/2}. \tag{4.15}$$

The speed of propagation of the fracture when $c_3/c_2 = 0.5$ and $V_0 = 1$ is

$$\frac{dL}{dt} = 0.036\frac{EH^3}{(1-\sigma^2)\mu BL_0}\left[1 + 0.071\frac{EH^3}{(1-\sigma^2)\mu BL_0^2}t\right]^{-1/2} . \tag{4.16}$$

The characteristic fluid velocity in the x-direction is obtained from (4.5):

$$U = \frac{PH^2}{\mu L_0}, \tag{4.17}$$

where P is the pressure at the fracture entry.

We now consider an application.

(i) Fracturing rock with ultra-high pressure water

A method has recently been proposed for fracturing rock in mines, namely using ultra-high pressure water to open fissures in the rock. Experimental evidence shows that pressures will lie in the range 250 to 1000 MPa and that fracturing takes place over a deflagration time scale of duration 0.5×10^{-3} to 2×10^{-3} s. We will assume that the pressure at the fracture entry is constant over the deflagration time scale. The parameter values are given in Table 2. The pressure dependence of the viscosity of water is anomalous since it decreases with increase in pressure [22]. From (4.15), the fracture length increases as the breadth decreases. Narrow fractures will propagate further than broad fractures. In order to estimate the maximum extent a fracture could propagate in the deflagration time scale we consider a narrow fracture with breadth equal to width: $B = 2H$. End effects will be neglected even although they are significant since B is small.

Consider first the lower bound, 250 MPa, for the pressure at the entry to the fracture with time scale 0.5×10^{-3} s. In Table 3 the results when $V_0 = 1$ are listed for a pre-existing fracture of length 50 cm. The fracture propagates to 79 cm after 0.5×10^{-3} s. Initially the propagation speed of the fracture is 760 ms^{-1} which decreases to 480 ms^{-1} after 0.5×10^{-3} s. From (4.17), the characteristic fluid velocity is 140 ms^{-1}. The assumptions of lubrication theory are satisfied but $\mathrm{Re}^* = 1.65 \times 10^4$ which is in the turbulent flow regime. The fluid velocity in the fracture will decrease as the fracture propagates and from (4.17) it will have reduced to about 90 ms^{-1} when the length of the fracture is about 80 cm.

Finally, consider the upper bound, 1000 MPa, for the pressure at the fracture entry with time scale 2×10^{-3} s. A pre-existing fracture of length 50 cm is again considered and the results when $V_0 = 1$ are listed in Table 3. The length of the fracture after 2×10^{-3} s is 1.33 m. The initial speed of propagation of the fracture is 760 ms^{-1} and the final speed is 286 ms^{-1}. From (4.17), the characteristic fluid velocity is now 575 ms^{-1}. This yields $\varepsilon^2\mathrm{Re} = 0.33$ which is approaching the limit beyond which lubrication theory ceases to be valid. Since $\mathrm{Re}^* = 6.6 \times 10^4$ the flow is in the turbulent regime. From (4.17), the fluid speed will have decreased to about 215 ms^{-1} when the fracture length is 1.33 m.

This preliminary analysis shows that for a fracture to propagate to a significant extent its breadth must be small, of order of magnitude the width of the fracture.

This appears to indicate that the process would not break the rock into blocks The process would result in the strength of the rock being reduced due to the propagation of microcracks.

The results are the same order of magnitude as reported by Meglis et. al. [23] when investigating the damage induced during excavation of a test tunnel in granite. They found microcracks as deep as 1m from the tunnel wall.

For the parameters used the fluid flow in the fracture is turbulent The theory of turbulent fluid fracture should be applied [2].

5. Concluding remarks

By adopting the PKN elasticity hypothesis, and using the Lie point symmetries of the resulting nonlinear diffusion equation we were able to derive a similarity solution for a fluid-driven fracture in rock and to reformulate the boundary value problem as a pair of initial value problems The pair of initial value problems were easier to solve than the original boundary value problem.

The similarity solution has several useful features. It describes the fluid-driven propagation of a pre-existing fracture. The role of pre-existing fractures may be key to deciding whether or not hydraulic fracturing is viable as a means of fracturing rock in mining. The solution contains an undetermined parameter, c_3/c_2, which can be chosen to impose a range of operating conditions at the entry to the fracture. The two important conditions, constant rate of fluid injection into the fracture and constant pressure at the fracture entry, can be imposed. Other operating conditions can be considered. For example, if the rate of working of the pressure at the entry to the fracture is independent of time, which may apply if fluid is injected by a pump working at a constant rate, then

$$p(0,t)\frac{dV(t)}{dt} = \text{constant}. \tag{5.1}$$

Condition (5.1) is satisfied provided $c_3/c_2 = 5/7 = 0.714$.

In the PKN model, the fluid pressure is readily derived since it is proportional to the half-width h. An advantage is that conditions based on pressure at the fracture entry can be easily imposed on the similarity solution. However, the pressure at the fracture tip is necessarily zero and therefore no stress intensity factor can be defined in a meaningful manner. Exponents of the PKN model have developed various ways of dealing with this shortcoming, but it remains an underlying difficulty of the PKN model

At the extreme conditions of short time scales as may exist, for example, in the proposed mechanism for fracturing rock by ultra-high pressure water, it is possible that standard lubrication theory may no longer be appropriate and instead the "impulsive" lubrication theory equations, where a u_t term is included in the momentum balance equation, should be used. The conditions for impulsive lubrication theory to apply can easily be derived: in addition to $\varepsilon \ll 1$ and $\varepsilon^2\mathrm{Re} \ll 1$, we find that the u_t term should be included whenever

$$H^2 \geq \tau\nu, \tag{5.2}$$

where τ is the time scale If these circumstances do pertain, however, the problem is significantly more complicated

Extensions of the group invariant solution could be addressed in future studies Fluid leak-off at the fracture tip and at locations on the fracture boundary was not included and can be important [24]. At the ultra-high pressures considered in rock fracturing the fluid may slip on the fracture boundary [25] and the dependence of viscosity on pressure may be significant [26]. The extension to turbulent fluid fracture should be considered.

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